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On Generalized $\mathcal{T}$-Curvature Tensor of Para-Kenmotsu Manifolds

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***Abstract:*--The object of the present paper is to generalize $\mathcal{T}$-curvature tensor of para-Kenmotsu manifold with the help of a new generalized (0,2) symmetric tensor $\mathcal{Z}$ introduced by Mantica and Suh [7]. It is shown that a generalized $\mathcal{T}\phi$-symmetric para-Kenmotsu manifold is an Einstein manifold.**

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***Keywords and phrases:*-- $\mathcal{T}$-curvature tensor, para-Kenmotsu manifold, Einstein manifold, η-Einstein manifold, Generalized $\mathcal{T}$-curvature tensor.**

I. INTRODUCTION

Several years ago, the notion of paracontact metric structures were introduced in [4].

Since the publication of [15], paracontact metric manifolds have been studied by many authors in recent years. The importance of para-Kenmotsu geometry, have been pointed out especially in the last years by several papers highlighting the exchanges with the theory of para-K$\ddot{a}$hler manifolds and its role in semi-Riemannian geometry and mathematical physics [3, 5, 6, 11, 8].

Tripathi and Gupta [14] had developed the notion of $\mathcal{T}$-curvature tensor in pseudo-Riemannian manifolds. They defined $\mathcal{T}$-curvature tensor as follows.

Definition 1.1 In a n-dimensional pseudo-Riemannian manifold (M, g), a $\mathcal{T}$-curvature tensor is a tensor of type (1,3) defined by

$$\begin{aligned}\mathcal{T}(X,Y,Z) = \; & c_0R(X,Y,Z) + c_1S(Y,Z)X + c_2S(X,Z)Y \\ & + c_3S(X,Y)Z + c_4g(Y,Z)QX + c_5g(X,Z)QY \\ & + c_6g(X,Y)QZ + rc_7[g(Y,Z)X - g(X,Z)Y],\end{aligned} \tag{1.1}$$

where $X, Y, Z \in \mathfrak{X}(M)$; $c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7$ are smooth functions on M, S, Q, R, r, g are respectively the Ricci tensor, Ricci operator, curvature tensor, scalar curvature and pseudo-Riemannian metric tensor.

Definition 1.2 The Riemannian curvature tensor R of type (0,4) on M is a quadri-linear mapping $R: \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow C^\infty(M)$ defined by $'R(X,Y,Z,W) = g(R(X,Y,Z),W)$ for any $X,Y,Z,W \in \mathfrak{X}(M)$

$\mathcal{T}$-curvature tensor reduces to many other curvature tensors for different values of $c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7$.

Definition 1.3 A $\mathcal{T}$-curvature tensor of type (0,4) is defined by

$$\begin{aligned}'\mathcal{T}(X,Y,Z,W) = \; & c_0\,'R(X,Y,Z,W) + c_1S(Y,Z)g(X,W) + c_2S(X,Z)g(Y,W) \\ & + c_3S(X,Y)g(Z,W) + c_4g(Y,Z)S(X,W) + c_5g(X,Z)S(Y,W) \\ & + c_6g(X,Y)S(Z,W) + rc_7[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)],\end{aligned} \tag{1.2}$$

where $X, Y, Z \in \mathfrak{X}(M)$, R is the Riemannian curvature tensor, S is the Ricci tensor, g is the pseudo-Riemannian metric tensor and $'T(X,Y,Z,W) = g(T(X,Y,Z),W)$.

In this paper, we consider the generalized $\mathcal{T}$ curvature tensor of para-Kenmotsu manifolds and study some properties of generalized $\mathcal{T}$ curvature tensor. The organisation of the paper is as follows:

After preliminaries on para-Kenmotsu manifold in section 2, we describe briefly the generalized $\mathcal{T}$ curvature tensor on para-Kenmotsu manifold in section 3 and also we study some properties of generalized $\mathcal{T}$ curvature tensor in para-Kenmotsu manifold. The last section is devoted to the study of the generalized $\mathcal{T}\phi$-symmetric para-Kenmotsu manifold is an Einstein manifold.

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II. PRELIMINARIES

The notion of an almost para-contact manifold was introduced by I. Sato [10].

An n-dimensional differentiable manifold M^n is said to have almost para-contact structure (ϕ, ξ, η), where ϕ is a tensor field of type (1,1), ξ is a vector field known as characteristic vector field and η is a 1-form satisfying the following relations

$$\phi^2(X) = X - \eta(X)\xi, \tag{2.1}$$

$$\eta(\phi X) = 0, \tag{2.2}$$

$$\phi(\xi) = 0, \tag{2.3}$$

and

$$\eta(\xi) = 1. \tag{2.4}$$

A differentiable manifold with almost para-contact structure (ϕ, ξ, η) is called an almost para-contact manifold. Further, if the manifold M^n has a semi-Riemannian metric g satisfying

$$\eta(X) = g(X, \xi) \tag{2.5}$$

and

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y). \tag{2.6}$$

Then the structure (ϕ, ξ, η, g) satisfying conditions (2.1) to (2.6) is called an almost para-contact Riemannian structure and the manifold M^n with such a structure is called an almost para-contact Riemannian manifold [1, 10].

On a para-Kenmotsu manifold [2, 11, 9], the following relations hold:

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.7}$$

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.8}$$

$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y), \tag{2.9}$$

$$\eta(R(X, Y, Z)) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \tag{2.10}$$

$$R(X, Y, \xi) = \eta(X)Y - \eta(Y)X, \tag{2.11}$$

$$R(X, \xi, Y) = -R(\xi, X, Y) = g(X, Y)\xi - \eta(Y)X, \tag{2.12}$$

$$S(\phi X, \phi Y) = -(n-1)g(\phi X, \phi Y), \tag{2.13}$$

$$S(X, \xi) = -(n-1)\eta(X), \tag{2.14}$$

$$Q\xi = -(n-1)\xi, \tag{2.15}$$

$$r = -n(n-1), \tag{2.16}$$

For any vector fields X, Y, Z , where Q is the Ricci operator that is $g(QX, Y) = S(X, Y)$, S is the Ricci tensor and r is the scalar curvature.

In [2], Blaga has given an example on para-Kenmotsu manifold:

A para-Kenmotsu manifold is said to be η-Einstein if its Ricci tensor S is of the form

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),$$

For arbitrary vector fields X and Y, where a and b are smooth functions on M^n.

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III. GENERALIZED $\mathcal{T}$-CURVATURE TENSOR OF PARA-KENMOTSU MANIFOLD

In this section, we give a brief account of generalized $\mathcal{T}$-curvature tensor of para-Kenmotsu manifold and studied various geometric properties of it.

The $\mathcal{T}$-curvature tensor is defined by Tripathi and Gupta

$$\begin{aligned} T(X,Y,Z) = \ & c_0R(X,Y,Z) + c_1S(Y,Z)X + c_2S(X,Z)Y \\ & +c_3S(X,Y)Z + c_4g(Y,Z)QX + c_5g(X,Z)QY \\ & +c_6g(X,Y)QZ + rc_7[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{3.1}$$

such a tensor field $\mathcal{T}$ is known as $\mathcal{T}$-curvature tensor.

Also, the type $(0,4)$ tensor field $'\mathcal{T}$ is given by

$$\begin{aligned} 'T(X,Y,Z,W) = \ & c_0\,'R(X,Y,Z,W) + c_1S(Y,Z)g(X,W) + c_2S(X,Z)g(Y,W) \\ & +c_3S(X,Y)g(Z,W) + c_4g(Y,Z)S(X,W) + c_5g(X,Z)S(Y,W) \\ & +c_6g(X,Y)S(Z,W) + rc_7[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)], \end{aligned} \tag{3.2}$$

where

$$'\mathcal{T}(X,Y,Z,W) = g(\mathcal{T}(X,Y,Z),W)$$

and

$$'R(X,Y,Z,W) = g(R(X,Y,Z),W)$$

for the arbitrary vector fields X,Y,Z,W.

Differentiating covariantly equation (3.1) with respect to P, we get

$$\begin{aligned} (\nabla_P\mathcal{T})(X,Y)Z) = \ & c_0(\nabla_PR)(X,Y)Z) + c_1(\nabla_PS)(Y,Z)X + c_2(\nabla_PS)(X,Z)Y \\ & +c_3(\nabla_PS)(X,Y)Z + c_4g(Y,Z)(\nabla_PQ)X \\ & +c_5g(X,Z)(\nabla_PQ)Y + c_6g(X,Y)(\nabla_PQ)Z \\ & +dr(P)c_7[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{3.3}$$

A new generalized $(0,2)$ symmetric tensor $\mathcal{Z}$ is defined by Mantica and Suh [7]

$$\mathcal{Z}(X,Y) = S(X,Y) + \psi g(X,Y), \tag{3.4}$$

where ψ is an arbitrary scalar function.

From equation (3.4), we have

$$\mathcal{Z}(\phi X,\phi Y) = S(\phi X,\phi Y) + \psi g(\phi X,\phi Y), \tag{3.5}$$

which on using equations (2.6) and (2.13), gives

$$\mathcal{Z}(\phi X,\phi Y) = [\psi - (n-1)][-g(X,Y) + \eta(X)\eta(Y)]. \tag{3.6}$$

From equation (3.4) in (3.2) equation reduces to

$$\begin{aligned} '\mathcal{T}(X,Y,Z,W) = \ & c_0\,'R(X,Y,Z,W) + c_1\mathcal{Z}(Y,Z)g(X,W) + c_2\mathcal{Z}(X,Z)g(Y,W) \\ & +c_3\mathcal{Z}(X,Y)g(Z,W) + c_4g(Y,Z)\mathcal{Z}(X,W) + c_5g(X,Z)\mathcal{Z}(Y,W) \\ & +c_6g(X,Y)\mathcal{Z}(Z,W) + rc_7[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ & -\psi[c_1g(Y,Z)g(X,W) + c_2g(X,Z)g(Y,W) + c_3g(X,Y)g(Z,W) \\ & +c_4g(Y,Z)g(X,W) + c_5g(X,Z)g(Y,W) + c_6g(X,Y)g(Z,W)]. \end{aligned} \tag{3.7}$$

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Let

$$'\mathcal{T}^*(X,Y,Z,W) = c_0\,'R(X,Y,Z,W) + c_1\mathcal{Z}(Y,Z)g(X,W) + c_2\mathcal{Z}(X,Z)g(Y,W) + c_3\mathcal{Z}(X,Y)g(Z,W) + c_4 g(Y,Z)\mathcal{Z}(X,W) + c_5 g(X,Z)\mathcal{Z}(Y,W) + c_6 g(X,Y)\mathcal{Z}(Z,W) + rc_7[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)], \quad (3.8)$$

In the above equation, we get

$$'\mathcal{T}^*(X,Y,Z,W) = {}'\mathcal{T}(X,Y,Z,W) + \psi[c_1 g(Y,Z)g(X,W) + c_2 g(X,Z)g(Y,W) + c_3 g(X,Y)g(Z,W) + c_4 g(Y,Z)g(X,W) + c_5 g(X,Z)g(Y,W) + c_6 g(X,Y)g(Z,W)]. \quad (3.9)$$

Thus $'\mathcal{T}^*$ defined in equation (3.8) is called generalized $\mathcal{T}$- curvature tensor of para-Kenmotsu manifold.

If ψ=0, then from eqauation (3.9), we have

$$'\mathcal{T}^*(X,Y,Z,W) = {}'\mathcal{T}(X,Y,Z,W). \quad (3.10)$$

Lemma 1 If the scalar function ψ vanishes on para-Kenmotsu manifold, then the $\mathcal{T}$- curvature tensor and generalized $\mathcal{T}$-curvature tensor are identicle.

Lemma 2 Generalized $\mathcal{T}$ -curvature tensor of para-Kenmotsu manifold satisfies Bianchi's first identity.

Remark 1 Generalized $\mathcal{T}$ -curvature tensor $'\mathcal{T}^*$ of para-Kenmotsu manifold is

- skew symmetric in first two slots.
- skew symmetric in last two slots.
- symmetric in pair of slots.

Proposition 1 Generalized $\mathcal{T}$ -curvature tensor of para-Kenmotsu manifold satisfies the following identities:

$$(a)\mathcal{T}^*(\xi,Y,Z) = -\mathcal{T}^*(Y,\xi,Z) = c_0[\eta(Z)Y - g(Y,Z)\xi] + c_1[S(Y,Z) + \psi g(Y,Z)]\xi + c_2\eta(Z)Y[\psi-(n-1)] + c_3\eta(Y)Z[\psi-(n-1)] + c_4 g(Y,Z)\xi[\psi-(n-1)] + c_5\eta(Z)[QY + \psi Y] + c_6\eta(Y)[QZ+\psi Z] + rc_7[g(Y,Z)\xi - \eta(Z)Y], \quad (3.11)$$

$$(b)\mathcal{T}^*(X,Y,\xi) = c_0[\eta(X)Y - \eta(Y)X] + c_1\eta(Y)X[\psi-(n-1)] + c_2\eta(X)Y[\psi-(n-1)] + c_3[g(X,Y)\psi + S(X,Y)]\xi + c_4\eta(Y)[\psi X + QX] + c_5\eta(X)[\psi Y + QY] + c_6 g(X,Y)\xi[\psi-(n-1)] + rc_7[\eta(Y)X - \eta(X)Y], \quad (3.12)$$

$$(c)\ \eta(\mathcal{T}^*(X,Y,Z)) = c_0[g(X,Z)\eta(Y) - g(Z,Y)\eta(X)] + c_1\eta(X)[g(Z,Y)\psi + S(Z,Y)] + c_2\eta(Y)[g(Z,X)\psi + S(Z,X)] + c_3\eta(Z)[g(Y,X)\psi + S(Y,X)] + c_4\eta(X)g(Y,Z)[\psi-(n-1)] + c_5\eta(Y)g(X,Z)[\psi-(n-1)] + c_6\eta(Z)g(X,Y)[\psi-(n-1)] + rc_7[g(Y,Z)\eta(X) - g(Z,X)\eta(Y)]. \quad (3.13)$$

IV. Generalized $\mathcal{T}\phi$-Symmetric Para-Kenmotsu Manifold

Definition 4.1 A para-Kenmotsu manifold M^n is said to be locally ϕ-symmetric if

$$\phi^2((\nabla_P R)(X,Y,U)) = 0, \quad (4.1)$$

for all vector fields X,Y,U,P orthogonal to ξ.
This notion was introduced by Takahashi for Sasakian manifold [12].

Definition 4.2 A para-Kenmotsu manifold is said to be ϕ-symmetric if

$$\phi^2((\nabla_P R)(X,Y,U)) = 0, \quad (4.2)$$

For arbitrary vector fields X, Y, U, P.

This notion was also introduced by Takahashi for Sasakian manifold [13]. Also analogous to these definitons, we define

Definition 4.3 A para-Kenmotsu manifold M^n is said to be generalized $\mathcal{T}$ locally ϕ-symmetric para-Kenmotsu manifold if

$$\phi^2((\nabla_P \mathcal{T}^*)(X, Y, U)) = 0, \tag{4.3}$$

for all vector fields X, Y, U, P orthogonal to ξ.

And also

Definition 4.4 A para-Kenmotsu manifold M^n is said to be generalized $\mathcal{T}\phi$-symmetric para-Kenmotsu manifold if

$$\phi^2((\nabla_P \mathcal{T}^*)(X, Y, U)) = 0, \tag{4.4}$$

for arbitary vector fields X, Y, U, P.

Theorem 4.1 A generalized $\mathcal{T}\phi$-symmetric para Kenmotsu manifold is an Einstein manifold.

Proof. Taking covariant derivative of equation (3.9) with respect to vector field P, we obtain

$$\begin{aligned}(\nabla_P \mathcal{T}^*)(X, Y, Z) = \; & (\nabla_P \mathcal{T})(X, Y, Z) + dr(\psi)[c_1 g(Y, Z)X \\ & + c_2 g(X, Z)Y + c_3 g(X, Y)Z + c_4 g(Y, Z)X \\ & + c_5 g(X, Z)Y + c_6 g(X, Y)Z],\end{aligned} \tag{4.5}$$

Using equation (3.3) in the above equation, we yields

$$\begin{aligned}(\nabla_P \mathcal{T}^*)(X, Y, Z) = \; & c_0(\nabla_P R)(X, Y, Z) + c_1(\nabla_P S)(Y, Z)X + c_2(\nabla_P S)(X, Z)Y \\ & + c_3(\nabla_P S)(X, Y)Z + c_4 g(Y, Z)(\nabla_P Q)X + c_5 g(X, Z)(\nabla_P Q)Y \\ & + c_6 g(X, Y)(\nabla_P Q)Z + dr(P)c_7[g(Y, Z)X - g(X, Z)Y] \\ & + dr(\psi)[c_1 g(Y, Z)X + + c_2 g(X, Z)Y + c_3 g(X, Y)Z + c_4 g(Y, Z)X \\ & + c_5 g(X, Z)Y + c_6 g(X, Y)Z],\end{aligned} \tag{4.6}$$

Assume that the manifold is generalized $\mathcal{T}\phi$-symmetric, then from equation (4.4), we have

$$\phi^2((\nabla_P \mathcal{T}^*)(X, Y, Z)) = 0,$$

which on using equation (2.1), gives

$$(\nabla_P \mathcal{T}^*)(X, Y, Z) = \eta((\nabla_P \mathcal{T}^*)(X, Y, Z))\xi. \tag{4.7}$$

Using equation (4.6) in above equation, we get

$$\begin{aligned}& c_0(\nabla_P R)(X, Y, Z) + c_1(\nabla_P S)(Y, Z)X + c_2(\nabla_P S)(X, Z)Y \\ & + c_3(\nabla_P S)(X, Y)Z + c_4 g(Y, Z)(\nabla_P Q)X + c_5 g(X, Z)(\nabla_P Q)Y \\ & + c_6 g(X, Y)(\nabla_P Q)Z + dr(P)c_7[g(Y, Z)X - g(X, Z)Y] + dr(\psi)[c_1 g(Y, Z)X \\ & + c_2 g(X, Z)Y + c_3 g(X, Y)Z + c_4 g(Y, Z)X + c_5 g(X, Z)Y + c_6 g(X, Y)Z] \\ & = \eta((\nabla_P R)(X, Y, Z))\xi + c_1(\nabla_P S)(Y, Z)\eta(X)\xi + c_2(\nabla_P S)(X, Z)\eta(Y)\xi \\ & + c_3(\nabla_P S)(X, Y)\eta(Z)\xi + c_4 g(Y, Z)\eta((\nabla_P Q))\eta(X)\xi + c_5 g(X, Z) \\ & \eta((\nabla_P Q))\eta(Y)\xi + c_6 g(X, Y)\eta((\nabla_P Q))\eta(Z)\xi + dr(P)c_7 \\ & \xi + dr(\psi)[c_1 g(Y, Z)\eta(X) \\ & + c_2 g(X, Z)\eta(Y) + c_3 g(X, Y)\eta(Z) + c_4 g(Y, Z)\eta(X) \\ & + c_5 g(X, Z)\eta(Y) + c_6 g(X, Y)\eta(Z)]\end{aligned} \tag{4.8}$$

Taking inner product of the above equation with V, we get

$$\begin{aligned}&c_0 g((\nabla_P R)(X,Y,Z),V) + c_1(\nabla_P S)(Y,Z)g(X,V) + c_2(\nabla_P S)(X,Z)g(Y,V)\\&+c_3(\nabla_P S)(X,Y)g(Z,V) + c_4 g(Y,Z)g((\nabla_P Q)X,V)) + c_5 g(X,Z)g((\nabla_P Q)Y,V)\\&+c_6 g(X,Y)g((\nabla_P Q)Z,V)) + dr(P)c_7[g(Y,Z)g(X,V) - g(X,Z)g(Y,V)]\\&+dr(\psi)[c_1 g(Y,Z)g(X,V) + c_2 g(X,Z)g(Y,V) + c_3 g(X,Y)g(Z,V) + c_4 g(Y,Z)\\&g(X,V) + c_5 g(X,Z)g(Y,V) + c_6 g(X,Y)g(Z,V)] = \eta((\nabla_P R)(X,Y,Z))\eta(V)\\&+c_1(\nabla_P S)(Y,Z)\eta(X)\eta(V) + c_2(\nabla_P S)(X,Z)\eta(Y)\eta(V) + c_3(\nabla_P S)(X,Y)\\&\eta(Z)\eta(V) + c_4 g(Y,Z)\eta((\nabla_P Q))\eta(X)\eta(V) + c_5 g(X,Z)\eta((\nabla_P Q))\eta(Y)\eta(V)\\&+c_6 g(X,Y)\eta((\nabla_P Q))\eta(Z)\eta(V) + dr(P)c_7[g(Y,Z)\eta(X)\eta(V) - g(X,Z)\\&\eta(Y)\eta(V)] + dr(\psi)[c_1 g(Y,Z)\eta(X)\eta(V) + c_2 g(X,Z)\eta(Y)\eta(V)\\&+c_3 g(X,Y)\eta(Z)\eta(V) + c_4 g(Y,Z)\eta(X)\eta(V) + c_5 g(X,Z)\eta(Y)\eta(V)\\&+c_6 g(X,Y)\eta(Z)\eta(V)]\end{aligned} \tag{4.9}$$

Putting $X = V = e_i$ and taking summation over i, we obtaion

$$\begin{aligned}&[c_0 + nc_1 + c_2 + c_3](\nabla_P S)(Y,Z) + c_4 g(Y,Z)g((\nabla_P Q)e_i,e_i))\\&+c_5 g((\nabla_P Q)Y,Z) + c_6 g((\nabla_P Q)Z,Y) + dr(P)c_7[ng(Y,Z) - \eta(Y)\eta(Z)]\\&+dr(\psi)[n(c_1+c_4)g(Y,Z) + (c_2+c_3+c_5+c_6)\eta(Y)\eta(Z)]\\&-\eta((\nabla_P R)(e_i,Y,Z))\eta(e_i) - c_1(\nabla_P S)(Y,Z) - c_2(\nabla_P S)(e_i,Z)\eta(Y)\\&-c_3(\nabla_P S)(e_i,Y)\eta(Z) - c_4 g(Y,Z)\eta((\nabla_P Q)) - c_5\eta(Y)\eta(Z)\eta((\nabla_P Q))\\&-c_6\eta(Y)\eta(Z)\eta((\nabla_P Q)) - dr(P)c_7[g(Y,Z) - \eta(Y)\eta(Z)]\\&-dr(\psi)[(c_1+c_4)g(Y,Z) + (c_2+c_3+c_5+c_6)\eta(Y)\eta(Z)] = 0,\end{aligned} \tag{4.10}$$

Taking $Z = \xi$ in the above equation, we have

$$\begin{aligned}&[(n-1)c_1 + c_3 + c_2 + c_0](\nabla_P S)(Y,\xi) + c_4 g((\nabla_P Q)e_i,e_i))\eta(Y)\\&+c_5\eta((\nabla_P Q)Y) + c_6\eta((\nabla_P Q)Y) + dr(P)c_7(n-1)\eta(Y)\\&+dr(\psi)\eta(Y)[n(c_1+c_4)] - \eta((\nabla_P R)(e_i,Y,\xi))\eta(e_i)\\&-c_2(\nabla_P S)(e_i,\xi)\eta(Y) - c_3(\nabla_P S)(e_i,Y)\\&-c_4\eta((\nabla_P Q))\eta(Y) - c_5\eta((\nabla_P Q))\eta(Y) - c_6\eta((\nabla_P Q))\eta(Y)\\&-dr(\psi)\eta(Y)(c_1+c_4) = 0.\end{aligned} \tag{4.11}$$

Now

$$\eta((\nabla_P R)(e_i,Y,\xi)\eta(e_i) = g((\nabla_P R)(e_i,Y,\xi),\xi)g(e_i,\xi). \tag{4.12}$$

Also

$$\begin{aligned}g((\nabla_P R)(e_i,Y,\xi),\xi) \;&= g(\nabla_P R(e_i,Y,\xi),\xi) - g(R(\nabla_P e_i,Y,\xi),\xi)\\&-g(R(e_i,\nabla_P Y,\xi),\xi) - g(R(e_i,Y,\nabla_P\xi),\xi).\end{aligned} \tag{4.13}$$

Since $\{e_i\}$ is an orthonormal basis, so $\nabla_X e_i = 0$ and using equation (2.11), we get

$$g(R(e_i,\nabla_P Y,\xi),\xi) = 0,$$

As

$$g(R(e_i,Y,\xi),\xi) + g(R(\xi,\xi,Y),e_i) = 0,$$

We have

$$g(\nabla_P R(e_i,Y,\xi),\xi) + g(R(e_i,Y,\xi),\nabla_P\xi) = 0,$$

Using this fact, we get

$$g((\nabla_P R)(e_i,Y,\xi),\xi) = 0. \tag{4.14}$$

Using equation (4.14) in (4.11), we have

$$\begin{aligned}[(n-1)c_1 + c_3 + c_2 + c_0](\nabla_P S)(Y,\xi) = \;&(1-n)c_7 dr(P)\eta(Y)\\&-(1-n)(c_1+c_4)dr(\psi)\eta(Y),\end{aligned} \tag{4.15}$$

Taking $Y=\xi$ in above equation and using equations (2.4) and (2.14), we get

$$dr(\psi)=\left[\frac{c_7}{c_1+c_4}\right]dr(P), \tag{4.16}$$

which shows that r is constant. Now we have

$$(\nabla_P S)(Y,\xi)=\nabla_P S(Y,\xi)-S(\nabla_P Y,\xi)-S(Y,\nabla_P \xi),$$

Then by using (2.8), (2.9), (2.14) in the above equation, it follows that

$$(\nabla_P S)(Y,\xi)=-S(Y,P)-(n-1)g(Y,P). \tag{4.17}$$

So from equation (4.15), (4.16) and (4.17), This shows that

$$S(Y,P)=-(n-1)g(Y,P),$$

if $[(n-1)c_1+c_3+c_2+c_0]\neq 0$, which shows that M^n is an Einstein manifold.

REFERENCES

[1] Adati, T. and Miyazava, T. (1977),On Para-Contact Riemannian manifolds, Tru Math., 13(2), 27-39.

[2] Blaga, A.M. (2015),η- Ricci Solitions on para- Kenmotsu manifolds, Balkan J. Geom. Appl., 20, 1-13.

[3] Cappelletti-Montano, B., Kupeli Erken, I. and Murathan, C. (2012),Nullity conditions in paracontact geometry, Differ. Geom. Appl. , 30, 665-693.

[4] Kaneyuki, S. and Williams, F. L. (1985),Almost paracontact and parahodge structure on manifolds, Nagoya Math. J., 99, 173-187.

[5] Kupeli Erken, I.,Yamabe solitons on three-dimensional normal almost paracontact metric manifolds, Periodica Math. Hungarica, DOI:10.1007/s10998-019-00303-3.

[6] Kupeli Erken, I. and Murathan, C. (2017),A complete study of three-dimensional paracontact (k, μ, ν) -spaces, Int. J. Geom. Methods Mod. Phys., https://doi.org/10.1142/S0219887817501067.

[7] Mantica, C. A., Suh, Y.J. (2012),Pseudo Z symmetric Riemannian manifolds with harmonic curvature tensors, Int. J. Geom. Meth. Mod. Phys. 9(1) , 1250004, 21 pages.

[8] Raghuwanshi, T., Pandey, S. K., Pandey, M. K. and Goyal, A. (2022), On generalized W_2 -curvature tensor of para-Kenmotsu manifolds, filomat, 36 no. 3, 741-752.

[9] Sato, I. (1976),On a structure similar to the almost contact structure, Tensor, (N.S.), 30, 219-224.

[10] Sardar Arpan and De, Uday Chand (2020),η- Ricci Solitions on para- Kenmotsu manifolds, Differential Geometry Dynamical Systems, Vol. 22, 218-228.

[11] Takahashi, Toshio (1969),Sasakian manifold with pseudo-Riemannian metric, Tohoku Math. J. 2(21), no. 2, 271-290.

[12] Takahashi, T. (1977),Sasakian ϕ-symmetric spaces, Tohoku Math. J., (2) 29(1), 91-113.

[13] Tripathi, M. M. and Gupta, P. (2011),$\mathcal{T}$-curvature tensor on a semi-Riemannian manifold, J. Adv. Math. Studies, 4, 117-129.

[14] Zamkovoy, S. (2009),Canonical connections on paracontact manifolds, Ann. Glob. Anal. Geom., 36, 37-60.