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Ricci Semi-Symmetric P-Sasakian Manifold

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***Abstract*--In this paper, we have extended the study of projective semi-symmetric connections on the para-contact manifold. We study the curvature conditions of Ricci semi-symmetric type on a P-Sasakian manifold admitting a projective semi-symmetric non metric connection.**

***Keywords and phrases:* Projective semi-symmetric connection, P-Sasakian manifold, Einstein manifold, curvature tensor, para-contact manifold**

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I. INTRODUCTION

The study of semi-symmetric connections has been a very attractive field for investigations in the past many decades. Semi-symmetric connection was introduced by A. Friedmann and J. A. Schouten [6] in 1924. In 1930, E. Bartolotti [3] extended a geometrical meaning to such a connection. Further, H. A. Hayden [7] studied a metric connection with torsion on Riemannian manifold. After a long gap, in 1970, the study of semi-symmetric connections was resumed by K. Yano [16]. In particular, he studied semi-symmetric metric connections. Afterwards several researchers have carried out the study of semi-symmetric connections in a variety of directions such as [4, 8, 10, 14, 15].

In, 2001 P. Zhao and H. Song [17] defined and studied a type of semi-symmetric connection on Riemannian manifold which is projectively equivalent to the Levi-Civita connection ∇, i.e., has the same geodesic curves as ∇. This was termed as projective semi-symmetric connection. The studies on projective semi-symmetric connections have been further extended by P. Zhao [18], S.K. Pal et.al. [11] and others.

In continuation to the previous studies, we consider the projective semi-symmetric connection on a P-Sasakian manifold. The organisation of the paper is as follows: After preliminaries on P-Sasakian manifold in section 2, we describe briefly the projective semi-symmetric connection in section 3. In section 4, we study a P-Sasakian manifold admitting a projective semi-symmetric connection and show that a P-Sasakian manifold satisfying the condition $\tilde{R}\cdot\tilde{S}=0$ is an Einstein manifold.

II. PRELIMINARIES

The notion of an (almost) para-contact manifold was introduced by I. Sato [12]. An n-dimensional differentiable manifold M is said to have almost para-contact structure (ϕ,ξ,η) where ϕ is a tensor field of type $(1,1)$, ξ is a vector field known as characteristic vector field and η is a 1-form satisfying the following relations

$$\phi^2(X) = X - \eta(X)\xi, \qquad (2.1)$$

$$\eta(\bar{X}) = 0, \qquad (2.2)$$

$$\phi(\xi) = 0, \qquad (2.3)$$

and

$$\eta(\xi) = 1. \qquad (2.4)$$

A differentiable manifold with almost para-contact structure (ϕ,ξ,η) is called an almost para-contact manifold. Further, if the M has a Riemannian metric g satisfying

$$\eta(X) = g(X,\xi), \qquad (2.5)$$

and

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y). \qquad (2.6)$$

Then the set (ϕ,ξ,η,g) satisfying the conditions (2.1) to (2.6) is called an almost para-contact Riemannian structure and the manifold M with such a structure is called an almost para-contact Riemannian manifold [2, 12].

Now, let (M,g) be an n-dimensional Riemannian manifold with a positive definite metric g admitting a $1-$form η which satisfies the conditions

$$(\nabla_X\eta)Y - (\nabla_Y\eta)X = 0 \qquad (2.7)$$

and

$$(\nabla_X\nabla_Y\eta)(Z) = -g(X,Z)\eta(Y) - g(X,Y)\eta(Z) + 2\eta(X)\eta(Y)\eta(Z), \qquad (2.8)$$

where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. Moreover, If (M,g) admits a vector field ξ and a $(1,1)$ tensor field ϕ such that

$$g(X,\xi) = \eta(X), \quad \eta(\xi) = 1 \quad and \quad \nabla_X\xi = \phi(X), \qquad (2.9)$$

Then it can be easily verified that the manifold under consideration becomes an almost paracontact Riemannian manifold. Such a manifold is called a para-Sasakian manifold or briefly a P-Sasakian manifold [1]. It is a special case of almost paracontact Riemannian manifold introduced by I. Sato. It is known [1] that on a P-Sasakian manifold the following relations hold:

$$\eta(R(X,Y,Z)) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \quad (2.10)$$

$$R(\xi,X,Y) = \eta(Y)X - g(X,Y)\xi, \quad (2.11)$$

$$R(\xi,X,\xi) = X - \eta(X)\xi, \quad (2.12)$$

$$R(X,Y,\xi) = \eta(X)Y - \eta(Y)X, \quad (2.13)$$

$$S(X,\xi) = -(n-1)\eta(X), \quad (2.14)$$

and

$$S(\phi X,\phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y), \quad (2.15)$$

where R is the curvature tensor and S is the Ricci tensor.

The tensors $R\cdot S$ are defined by the followings [9, 13]

$$(R(X,Y)\cdot S)(Z,W) = -S(R(X,Y,Z),W) - S(Z,R(X,Y,W)), \quad (2.16)$$

where $\cdot$ indicates that $R(X,Y)$ is acts as a derivation of the tensor algebra at each point of the manifold for tangent vectors X, Y. An n-dimensional manifold is said to be Ricci semi-symmetric if it satisfies $R\cdot S = 0$.

III. Projective Semi-Symmetric Connection

In this section, we give a brief account of projective semi-symmetric connection and study it on a P-Sasakian manifold.

A linear connection $\tilde{\nabla}$ on an n-dimensional Riemannian manifold (M,g) is called a semi-symmetric connection [16], if its torsion tensor T given by

$$T(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y]$$

has the form

$$T(X,Y) = \pi(Y)X - \pi(X)Y. \quad (3.1)$$

where π is a 1-form associated with a vector field ρ, i.e.,

$$\pi(X) = g(X,\rho). \quad (3.2)$$

Further, a connection $\tilde{\nabla}$ is a metric connection if it satisfies

$$(\tilde{\nabla}_X g)(Y,Z) = 0. \quad (3.3)$$

If the geodesic with respect to $\tilde{\nabla}$ are always consistent with those of the Levi-Civita connection ∇ on a Riemannian manifold, then $\tilde{\nabla}$ is called a connection projectively equivalent to ∇. If $\tilde{\nabla}$ is linear connection projective equivalent to ∇ as well as a semi-symmetric one, we call $\tilde{\nabla}$ is called projective semi-symmetric connection [17].

Now, we consider a projective semi-symmetric connection $\tilde{\nabla}$ introduced by P. Zhao and H. Song [17] given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \Psi(Y)X + \Psi(X)Y + \Phi(Y)X - \Phi(X)Y, \quad (3.4)$$

for arbitrary vector fields X and Y, where the 1-forms Ψ and Φ are given through the following relations:

$$\Psi(X) = \frac{n-1}{2(n+1)}\pi(X) \text{ and } \Phi(X) = \frac{1}{2}\pi(X). \quad (3.5)$$

It is easy to see that the equations (3.4) and (3.5) give us

$$(\tilde{\nabla}_X g)(Y,Z) = \frac{1}{n+1}[2\pi(X)g(Y,Z) - n\pi(Y)g(X,Z) - n\pi(Z)g(X,Y)], \quad (3.6)$$

which shows that the connection $\tilde{\nabla}$ given by (3.4) is a metric one.

We denote by $\tilde{R}$ and R the curvature tensors of the manifold relative to the projective semi-symmetric connection connections $\tilde{\nabla}$ and the Levi-Civita connection ∇. It is known that [17] that

$$\tilde{R}(X,Y,Z) = R(X,Y,Z) + \alpha(X,Z)Y - \alpha(Y,Z)X + \beta(X,Y)Z, \quad (3.7)$$

where α and β are the tensors of type $(0,2)$ given by the following relations

$$\alpha(X,Y) = \Psi'(X,Y) + \Phi'(Y,X) - \Psi(X)\Phi(Y) - \Psi(Y)\Phi(X) \quad (3.8)$$

$$\beta(X,Y) = \Psi'(X,Y) - \Psi'(Y,X) + \Phi'(Y,X) - \Phi'(X,Y). \tag{3.9}$$

The tensors Ψ' and Φ' of type $(0,2)$ are defined by the following two relations.

$$\Psi'(X,Y) = (\nabla_X \Psi)(Y) - \Psi(X)\Psi(Y). \tag{3.10}$$

and

$$\Phi'(X,Y) = (\nabla_X \Phi)(Y) - \Phi(X)\Phi(Y). \tag{3.11}$$

Contraction of the vector field X in the equation (3.7) yields a relation between Ricci tensors of the manifold relative to the two connections $\widetilde{\nabla}$ and ∇ which is given by

$$\tilde{S}(Y,Z) = S(Y,Z) + \beta(Y,Z) - (n-1)\alpha(Y,Z) \tag{3.12}$$

Also, from the above equation, we get the following equation relating scalar curvatures $\tilde{r}$ and r of the manifold with respect to the two connections $\widetilde{\nabla}$ and ∇

$$\tilde{r} = r + b - (n-1)a. \tag{3.13}$$

where $b = \sum_{i=1}^{n} \beta(e_i, e_i)$ and $a = \sum_{i=1}^{n} \alpha(e_i, e_i)$.

In order to extend the studies of the projective semi-symmetric connection $\widetilde{\nabla}$ on P-Sasakian manifold, we identify the 1-form π of the connection $\widetilde{\nabla}$ with the 1-form η of the P-Sasakian manifold. In view of this equality between π and η and the equations (3.5), we find that the expression (3.4) for the projective semi-symmetric connection $\widetilde{\nabla}$ reduces to

$$\widetilde{\nabla}_X Y = \nabla_X Y + \frac{1}{2}(c+1)\eta(Y)X + \frac{1}{2}(c-1)\eta(X)Y, \tag{3.14}$$

where the constant c is given by $c = \frac{n-1}{n+1}$. Now, it can be seen observed that

$$(\widetilde{\nabla}_X \eta)(Y) = (\nabla_X \eta)(Y) - c\eta(X)\eta(Y). \tag{3.15}$$

On a P-Sasakian manifold, we have $(\nabla_X \eta)(Y) = (\nabla_Y \eta)(X)$. Therefore, the above equation yields

$$(\widetilde{\nabla}_X \eta)Y = (\widetilde{\nabla}_Y \eta)X.$$

Thus, the connection $\widetilde{\nabla}$ given by the equation (3.4) becomes *special projective semi-symmetric connection* studied by S.K. Pal et. al. [11]. It can also be verified very easily that for such a projective semi-symmetric connection the tensor β vanishes and the tensor α is symmetric, i.e.,

$$\beta(X,Y) = 0, \quad \text{and} \quad \alpha(X,Y) = \alpha(Y,X). \tag{3.16}$$

As a consequence of these, the expressions for curvature tensor, the tensor α, Ricci tensors and scalar curvatures given by (3.7), (3.8), (3.12) and (3.13) takes the following simpler forms

$$\tilde{R}(X,Y,Z) = R(X,Y,Z) + \alpha(X,Z)Y - \alpha(Y,Z)X, \tag{3.17}$$

$$\alpha(X,Y) = \mu(\nabla_X \eta)(Y) - \mu^2\eta(X)\eta(Y), \tag{3.18}$$

$$\tilde{S}(Y,Z) = S(Y,Z) - (n-1)\alpha(Y,Z). \tag{3.19}$$

And

$$\tilde{r} = r - (n-1)a, \tag{3.20}$$

where $\mu = \frac{1}{2}(c+1)$.

It may also be notes that the Ricci tensor $\tilde{S}(Y,Z)$ of the special projective semi-symmetric connection is symmetric.

Now, we derive some of the results concerning the tensor α which we shall need in subsequent sections.

Replacing X by ξ in the equation (3.18) and using (2.4), we get

$$\alpha(\xi,Y) = \mu(\nabla_\xi\eta)(Y) - \mu^2\eta(Y). \tag{3.21}$$

Next, covariant differentiation of the equation (2.4) with respect to the Levi-Civita connection ∇ gives

$$(\nabla_X\eta)(\xi) + \eta(\nabla_X\xi) = 0,$$

which, in view of the equations (2.2), (2.7) and (2.9), yields

$$(\nabla_\xi\eta)(X) = (\nabla_X\eta)(\xi) = 0.$$

Using the above equation and the fact that the tensor α is symmetric in the above equation (3.21), we get

$$\alpha(\xi,Y) = \alpha(Y,\xi) = \lambda\eta(Y). \tag{3.22}$$

where we have put $\lambda = -\mu^2$.

Now, putting $X = \xi$ in the equation (3.17) and using the equations (2.11) and (3.22), we obtain

$$\tilde{R}(\xi,Y,Z) = \lambda'\eta(Z)Y - \theta(Y,Z)\xi, \tag{3.23}$$

where $\lambda' = (1+\lambda)$ and the tensor θ is a symmetric tensor given by

$$\theta(Y,Z) = g(Y,Z) + \alpha(Y,Z). \tag{3.24}$$

Again, taking $Z = \xi$ in the above equation and using the equation (3.22), we have

$$\theta(Y,\xi) = \lambda'\eta(Y). \tag{3.25}$$

IV. Ricci Semi-Symmetric P-Sasakian Manifold

In this section, we consider a P-Sasakian manifold which admits a projective semi-symmetric connection $\tilde{\nabla}$ and is Ricci Semmi-symmetric with respect the connection $\tilde{\nabla}$, i.e., satisfies the condition of the type $\tilde{R}\cdot\tilde{S} = 0$.

Theorem 4.1 If a P-Sasakian manifold admitting a projective semi-symmetric connection $\tilde{\nabla}$ is Ricci semi-symmetric with respect to the connection $\tilde{\nabla}$ then the manifold is an Einstein manifold.

Proof: Let the projective semi-symmetric connection $\tilde{\nabla}$ on P-Sasakian manifold satisfies

$$\left(\tilde{R}(X,Y)\cdot\tilde{S}\right)(Z,W) = 0. \tag{4.1}$$

where $\tilde{R}$ and $\tilde{S}$ are the curvature tensor and the Ricci tensor of the manifold relative to the connection $\tilde{\nabla}$. The equation (2.17) gives

$$-\tilde{S}(\tilde{R}(X,Y,Z),W) - \tilde{S}(Z,\tilde{R}(X,Y,W)) = 0.$$

Taking $X = \xi$ in the above equation, it follows that

$$\tilde{S}(\tilde{R}(\xi,Y,Z),W) + \tilde{S}(Z,\tilde{R}(\xi,Y,W)) = 0,$$

which, in view of the equation (3.23), gives

$$\lambda'\eta(Z)\tilde{S}(Y,W) - \theta(Y,Z)\tilde{S}(\xi,W) + \lambda'\eta(W)\tilde{S}(Z,Y) - \theta(Y,W)\tilde{S}(Z,\xi) = 0. \tag{4.2}$$

Replacing $Z = \xi$ in the equation (3.19) and using the equation (2.14) and (3.22), we obtain

$$\tilde{S}(Y,\xi) = d\eta(Y), \tag{4.3}$$

where $d = -(n-1)\lambda'$.

Now, putting $W = \xi$ in the equation (4.2) and using the equation (4.3), we get

$$\lambda'\tilde{S}(Y,Z) = d\,\theta(Y,Z).$$

Now, in view of the equations (3.19) and (3.24), we obtain

$$S(Y,Z) = -(n-1)g(Y,Z). \tag{4.4}$$

Thus, the manifold is Einstein manifold. This proves the theorem.

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