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On a Type of Projective Semi-Symmetric Connection

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***Abstract*-- This paper is devoted to the study of projective semi-symmetric connections on the para-contact manifold. We study the curvature conditions of $\tilde{R}\cdot\tilde{R}=0$ type on a P-Sasakian manifold admitting a projective semi-symmetric non metric connection.**

***Keywords and phrases:* Projective semi-symmetric connection, P-Sasakian manifold, Einstein manifold, curvature tensor, para-contact manifold**

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I. Introduction

The study of semi-symmetric connections has been a very attractive field for investigations in the past many decades. Semi-symmetric connection was introduced by A. Friedmann and J. A. Schouten [6] in 1924. In 1930, E. Bartolotti [3] extended a geometrical meaning to such a connection. Further, H. A. Hayden [7] studied a metric connection with torsion on Riemannian manifold. After a long gap, in 1970, the study of semi-symmetric connections was resumed by K. Yano [16]. In particular, he studied semi-symmetric metric connections. Afterwards several researchers have carried out the study of semi-symmetric connections in a variety of directions such as [4, 8, 10, 14, 15].

In, 2001 P. Zhao and H. Song [17] defined and studied a type of semi-symmetric connection on Riemannian manifold which is projectively equivalent to the Levi-Civita connection ∇, i.e., has the same geodesic curves as ∇. This was termed as projective semi-symmetric connection. The studies on projective semi-symmetric connections have been further extended by P. Zhao [18], S.K. Pal et.al. [11] and others.

Here, we consider the projective semi-symmetric connection on a P-Sasakian manifold. The organisation of the paper is as follows: After preliminaries on P-Sasakian manifold in section 2, we describe briefly the projective semi-symmetric connection in section 3. Further, in the section 4, we show that the condition $\tilde{R}\cdot\tilde{R}=0$ implies that the P-Sasakian manifold is an Einstein manifold under the additional assumption that 1-form η is recurrent.

II. Preliminaries

The notion of an (almost) para-contact manifold was introduced by I. Sato [12]. An n-dimensional differentiable manifold M is said to have almost para-contact structure (ϕ,ξ,η) where ϕ is a tensor field of type (1,1), ξ is a vector field known as characteristic vector field and η is a 1-form satisfying the following relations

$$\phi^2(X)=X-\eta(X)\xi, \tag{2.1}$$

$$\eta(\bar{X})=0, \tag{2.2}$$

$$\phi(\xi)=0, \tag{2.3}$$

and

$$\eta(\xi)=1. \tag{2.4}$$

A differentiable manifold with almost para-contact structure (ϕ,ξ,η) is called an almost para-contact manifold. Further, if the M has a Riemannian metric g satisfying

$$\eta(X)=g(X,\xi), \tag{2.5}$$

and

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y). \tag{2.6}$$

Then the set (ϕ,ξ,η,g) satisfying the conditions (2.1) to (2.6) is called an almost para-contact Riemannian structure and the manifold M with such a structure is called an almost para-contact Riemannian manifold [2, 12].

Now, let (M,g) be an n-dimensional Riemannian manifold with a positive definite metric g admitting a $1-$form η which satisfies the conditions

$$(\nabla_X\eta)Y-(\nabla_Y\eta)X=0 \tag{2.7}$$

and

$$(\nabla_X\nabla_Y\eta)(Z)=-g(X,Z)\eta(Y)-g(X,Y)\eta(Z)+2\eta(X)\eta(Y)\eta(Z), \tag{2.8}$$

where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. Moreover, If (M,g) admits a vector field ξ and a (1,1) tensor field ϕ such that

$$g(X,\xi)=\eta(X),\quad \eta(\xi)=1 \quad and \quad \nabla_X\xi=\phi(X), \tag{2.9}$$

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Then it can be easily verified that the manifold under consideration becomes an almost paracontact Riemannian manifold. Such a manifold is called a para-Sasakian manifold or briefly a P-Sasakian manifold [1]. It is a special case of almost paracontact Riemannian manifold introduced by I. Sato. It is known [1] that on a P-Sasakian manifold the following relations hold:

$$\eta(R(X,Y,Z)) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \quad (2.10)$$

$$R(\xi,X,Y) = \eta(Y)X - g(X,Y)\xi, \quad (2.11)$$

$$R(\xi,X,\xi) = X - \eta(X)\xi, \quad (2.12)$$

$$R(X,Y,\xi) = \eta(X)Y - \eta(Y)X, \quad (2.13)$$

$$S(X,\xi) = -(n-1)\eta(X), \quad (2.14)$$

and

$$S(\phi X,\phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y), \quad (2.15)$$

where R is the curvature tensor and S is the Ricci tensor.

The tensors $R\cdot R$ are defined by the followings [9, 13]

$$\begin{aligned}(R(X,Y)\cdot R)(Z,W,U) &= R(X,Y,R(Z,W,U)) - R(R(X,Y,Z),W,U)\\ &\quad -R(Z,R(X,Y,W),U) - R(Z,W,R(X,Y,U))\end{aligned}$$
(2.16)

where $\cdot$ indicates that $R(X,Y)$ is acts as a derivation of the tensor algebra at each point of the manifold for tangent vectors X, Y. An n-dimensional manifold is said to be semi-symmetric manifold if the tensor $R(X,Y)\cdot R = 0$.

III. Projective Semi-Symmetric Connection

In this section, we give a brief account of projective semi-symmetric connection and study it on a P-Sasakian manifold.

A linear connection $\widetilde{\nabla}$ on an n-dimensional Riemannian manifold (M,g) is called a semi-symmetric connection [16], if its torsion tensor T given by

$$T(X,Y) = \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X,Y]$$

has the form

$$T(X,Y) = \pi(Y)X - \pi(X)Y. \quad (3.1)$$

where π is a 1-form associated with a vector field ρ, i.e.,

$$\pi(X) = g(X,\rho). \quad (3.2)$$

Further, a connection $\widetilde{\nabla}$ is a metric connection if it satisfies

$$(\widetilde{\nabla}_X g)(Y,Z) = 0. \quad (3.3)$$

If the geodesic with respect to $\widetilde{\nabla}$ are always consistent with those of the Levi-Civita connection ∇ on a Riemannian manifold, then $\widetilde{\nabla}$ is called a connection projectively equivalent to ∇. If $\widetilde{\nabla}$ is linear connection projective equivalent to ∇ as well as a semi-symmetric one, we call $\widetilde{\nabla}$ is called projective semi-symmetric connection [17].

Now, we consider a projective semi-symmetric connection $\widetilde{\nabla}$ introduced by P. Zhao and H. Song [17] given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \Psi(Y)X + \Psi(X)Y + \Phi(Y)X - \Phi(X)Y, \quad (3.4)$$

for arbitrary vector fields X and Y, where the 1-forms Ψ and Φ are given through the following relations:

$$\Psi(X) = \frac{n-1}{2(n+1)}\pi(X) \text{ and } \Phi(X) = \frac{1}{2}\pi(X). \quad (3.5)$$

It is easy to see that the equations (3.4) and (3.5) give us

$$(\widetilde{\nabla}_X g)(Y,Z) = \frac{1}{n+1}[2\pi(X)g(Y,Z) - n\pi(Y)g(X,Z) - n\pi(Z)g(X,Y)], \quad (3.6)$$

which shows that the connection $\widetilde{\nabla}$ given by (3.4) is a metric one.

We denote by $\tilde{R}$ and R the curvature tensors of the manifold relative to the projective semi-symmetric connection connections $\widetilde{\nabla}$ and the Levi-Civita connection ∇. It is known that [17] that

$$\tilde{R}(X,Y,Z) = R(X,Y,Z) + \alpha(X,Z)Y - \alpha(Y,Z)X + \beta(X,Y)Z, \quad (3.7)$$

where α and β are the tensors of type $(0,2)$ given by the following relations

$$\alpha(X,Y) = \Psi'(X,Y) + \Phi'(Y,X) - \Psi(X)\Phi(Y) - \Psi(Y)\Phi(X) \quad (3.8)$$

$$\beta(X,Y) = \Psi'(X,Y) - \Psi'(Y,X) + \Phi'(Y,X) - \Phi'(X,Y). \tag{3.9}$$

The tensors Ψ' and Φ' of type $(0,2)$ are defined by the following two relations.

$$\Psi'(X,Y) = (\nabla_X\Psi)(Y) - \Psi(X)\Psi(Y). \tag{3.10}$$

and

$$\Phi'(X,Y) = (\nabla_X\Phi)(Y) - \Phi(X)\Phi(Y). \tag{3.11}$$

Contraction of the vector field X in the equation (3.7) yields a relation between Ricci tensors of the manifold relative to the two connections $\widetilde{\nabla}$ and ∇ which is given by

$$\tilde{S}(Y,Z) = S(Y,Z) + \beta(Y,Z) - (n-1)\alpha(Y,Z) \tag{3.12}$$

Also, from the above equation, we get the following equation relating scalar curvatures $\tilde{r}$ and r of the manifold with respect to the two connections $\widetilde{\nabla}$ and ∇

$$\tilde{r} = r + b - (n-1)a. \tag{3.13}$$

where $b = \sum_{i=1}^{n} \beta(e_i, e_i)$ and $a = \sum_{i=1}^{n} \alpha(e_i, e_i)$.

In order to extend the studies of the projective semi-symmetric connection $\widetilde{\nabla}$ on P-Sasakian manifold, we identify the 1-form π of the connection $\widetilde{\nabla}$ with the 1-form η of the P-Sasakian manifold. In view of this equality between π and η and the equations (3.5), we find that the expression (3.4) for the projective semi-symmetric connection $\widetilde{\nabla}$ reduces to

$$\widetilde{\nabla}_X Y = \nabla_X Y + \frac{1}{2}(c+1)\eta(Y)X + \frac{1}{2}(c-1)\eta(X)Y, \tag{3.14}$$

where the constant c is given by $c = \frac{n-1}{n+1}$. Now, it can be seen observed that

$$(\widetilde{\nabla}_X\eta)(Y) = (\nabla_X\eta)(Y) - c\eta(X)\eta(Y). \tag{3.15}$$

On a P-Sasakian manifold, we have $(\nabla_X\eta)(Y) = (\nabla_Y\eta)(X)$. Therefore, the above equation yields

$$(\widetilde{\nabla}_X\eta)Y = (\widetilde{\nabla}_Y\eta)X.$$

Thus, the connection $\widetilde{\nabla}$ given by the equation (3.4) becomes *special projective semi-symmetric connection* studied by S.K. Pal et. al. [11]. It can also be verified very easily that for such a projective semi-symmetric connection the tensor β vanishes and the tensor α is symmetric, i.e.,

$$\beta(X,Y) = 0, \quad \text{and} \quad \alpha(X,Y) = \alpha(Y,X). \tag{3.16}$$

As a consequence of these, the expressions for curvature tensor, the tensor α, Ricci tensors and scalar curvatures given by (3.7), (3.8), (3.12) and (3.13) takes the following simpler forms

$$\tilde{R}(X,Y,Z) = R(X,Y,Z) + \alpha(X,Z)Y - \alpha(Y,Z)X, \tag{3.17}$$

$$\alpha(X,Y) = \mu(\nabla_X\eta)(Y) - \mu^2\eta(X)\eta(Y), \tag{3.18}$$

$$\tilde{S}(Y,Z) = S(Y,Z) - (n-1)\alpha(Y,Z). \tag{3.19}$$

and

$$\tilde{r} = r - (n-1)a, \tag{3.20}$$

where $\mu = \frac{1}{2}(c+1)$.

It may also be notes that the Ricci tensor $\tilde{S}(Y,Z)$ of the special projective semi-symmetric connection is symmetric.

Now, we derive some of the results concerning the tensor α which we shall need in subsequent sections.

Replacing X by ξ in the equation (3.18) and using (2.4), we get

$$\alpha(\xi,Y) = \mu\big(\nabla_\xi \eta\big)(Y) - \mu^2\eta(Y). \tag{3.21}$$

Next, covariant differentiation of the equation (2.4) with respect to the Levi-Civita connection ∇ gives

$$(\nabla_X\eta)(\xi) + \eta(\nabla_X\xi) = 0,$$

which, in view of the equations (2.2), (2.7) and (2.9), yields

$$(\nabla_\xi\eta)(X) = (\nabla_X\eta)(\xi) = 0.$$

Using the above equation and the fact that the tensor α is symmetric in the above equation (3.21), we get

$$\alpha(\xi,Y) = \alpha(Y,\xi) = \lambda\eta(Y). \tag{3.22}$$

where we have put $\lambda = -\mu^2$.

Now, putting $X = \xi$ in the equation (3.17) and using the equations (2.11) and (3.22), we obtain

$$\tilde{R}(\xi,Y,Z) = \lambda'\eta(Z)Y - \theta(Y,Z)\xi, \tag{3.23}$$

where $\lambda' = (1+\lambda)$ and the tensor θ is a symmetric tensor given by

$$\theta(Y,Z) = g(Y,Z) + \alpha(Y,Z). \tag{3.24}$$

Again, taking $Z = \xi$ in the above equation and using the equation (3.22), we have

$$\theta(Y,\xi) = \lambda'\eta(Y). \tag{3.25}$$

IV. Semi-Symmetric P-Sasakian Manifold

In this section, we consider a P-Sasakian manifold admitting a projective semi-symmetric connection $\tilde{\nabla}$ whose 1-form η is a recurrent. More precisely, we suppose that the 1-form η of the manifold satisfies the following condition

$$(\nabla_X\eta)(Y) = \eta(X)\eta(Y). \tag{4.1}$$

Now, in view of this assumption, the equation (3.18) takes the following form

$$\alpha(X,Y) = \lambda\eta(X)\eta(Y). \tag{4.2}$$

Further, putting $Y = \xi$ in the above equation, we obtain

$$\alpha(X,\xi) = \lambda\eta(X).$$

Also, by taking inner product with ξ, we get the following from the equation (3.17)

$$\eta(\tilde{R}(X,Y,Z) = \eta(R(X,Y,Z) + \alpha(X,Z)\eta(Y) - \alpha(Y,Z)\eta(X), \tag{4.3}$$

which, due to the equations (2.10) and (4.2), gives

$$\eta(\tilde{R}(X,Y,Z) = \eta(R(X,Y,Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X). \tag{4.4}$$

Now, we state the following theorem.

Theorem 5.1 If a P-Sasakian manifold admitting a projective semi-symmetric connection $\tilde{\nabla}$ is semi-symmetric with respect to $\tilde{\nabla}$ and the 1-form η is recurrent then the manifold is an Einstein manifold.

Proof: Let the 1-form η satisfies the condition (4.1) and is semi-symmetric with respect to the connection $\tilde{\nabla}$, i.e., satisfies the condition

$$(\tilde{R}(X,Y)\cdot\tilde{R})(Z,W,U) = 0 \tag{5.5}$$

for all vector fields X, Y, Z, W and U. Then, in particular for $X = \xi$, we have

$$(\tilde{R}(\xi,X)\cdot\tilde{R})(Y,Z,W) = 0,$$

which, in view of the equation (2.16), implies that

$$\begin{aligned} 0 &= \tilde{R}(\xi,X,\tilde{R}(Y,Z,W)) - \tilde{R}(\tilde{R}(\xi,X,Y),Z,W) \\ &\quad -\tilde{R}(Y,\tilde{R}(\xi,X,Z),W) - \tilde{R}(Y,Z,\tilde{R}(\xi,X,W)). \end{aligned}$$

Now, using (2.10), (3.23), (3.24) and (4.4) in the above equation, we find that

$$\begin{aligned} 0 &= \lambda'[g(Y,W)\eta(Z) - g(Z,W)\eta(Y)]X - [g(X,\tilde{R}(Y,Z,W) + \alpha(X,\tilde{R}(Y,Z,W)]\xi \\ &-\lambda'\eta(Y)\tilde{R}(X,Z,W) + \theta(X,Y)\tilde{R}(\xi,Z,W) - \lambda'\eta(Z)\tilde{R}(Y,X,W) \\ &+\theta(X,Z)\tilde{R}(Y,\xi,W) - \lambda'\eta(W)\tilde{R}(Y,Z,X) + \theta(X,W)\tilde{R}(Y,Z,\xi). \end{aligned}$$

Again, using the equations (3.22) and (3.23), in the above equation and taking inner product with ξ, we get

$$\begin{aligned} {}'\tilde{R}(Y,Z,W,X) &= \lambda' g(Y,W)\eta(Z)\eta(X) - \lambda' g(Z,W)\eta(Y)\eta(X) - \lambda\eta(X)\eta(\tilde{R}(Y,Z,W) \\ &-\lambda'\eta(Y)\eta(\tilde{R}(X,Z,W)) + \lambda'\eta(Z)\eta(W)\theta(X,Y) - \theta(X,Y)\theta(Z,W) \\ &-\lambda'\eta(Z)\eta(\tilde{R}(Y,X,W) - \lambda'\eta(Y)\eta(W)\theta(X,Z) + \theta(X,Z)\theta(Y,W) \\ &-\lambda'\eta(W)\eta(\tilde{R}(Y,Z,X)) + (1+\lambda)\eta(Y)\eta(Z)\theta(X,W) - (1+\lambda)\eta(Z)\eta(Y)\theta(X,W), \end{aligned}$$

where ${}'\tilde{R}(X,Y,Z,W) = g(\tilde{R}(X,Y,Z),W)$ is the curvature tensor of type (0,4) relative to the connection $\tilde{\nabla}$.

Using (2.10) and (4.4) in the above equation, it follows that

$$\begin{aligned} {}'\tilde{R}(Y,Z,W,X) &= -\lambda\eta(X)g(Y,W)\eta(Z) + \lambda\eta(X)g(Z,W)\eta(Y) \\ &+\lambda'\eta(Z)\eta(W)\theta(X,Y) - \theta(X,Y)\theta(Z,W) \\ &-\lambda'\eta(Y)\eta(W)\theta(X,Z) + \theta(X,Z)\theta(Y,W) \\ &-\lambda'\eta(W)\eta(Z)g(Y,X) + \lambda'\eta(W)\eta(Y)g(Z,X). \end{aligned}$$

Using the equations (3.24) and (4.2) in the above equation, we find

$$\begin{aligned} {}'\tilde{R}(Y,Z,W,X) &= -g(X,Y)g(Z,W) - \lambda\eta(Z)\eta(W)g(X,Y) \\ &+g(X,Z)g(Y,W) + \lambda\eta(Y)\eta(W)g(X,Z). \end{aligned}$$

Now, contracting the above equation with respect to X and Y, we obtain

$$\tilde{S}(Z,W) = (1-n)g(Z,W) + (1-n)\lambda\eta(W)\eta(Z).$$

Also we may write

$$\tilde{S}(Y,Z) = (1-n)g(Y,Z) + (1-n)\lambda\eta(Y)\eta(Z),$$

which, on using the equation (3.19), gives

$$S(Y,Z) = (1-n)g(Y,Z).$$

This proves the theorem.

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