International Journal of Recent Development in Engineering and Technology
Website: www.ijrdet.com (ISSN 2347 - 6435 (Online)) Volume 4, Issue 9, September 2015)

Estimation of the Optimum Rotational Parameter for the Fractional Fourier Transform Using Domain Decomposition

Seema Sud[1]

[1]*The Aerospace Corporation, 4851 Stonecroft Blvd. Chantilly, VA 20151*

***Abstract*— The Fractional Fourier Transform (FrFT) provides significant interference suppression over the Fast Fourier Transform (FFT) when the signal-of-interest (SOI) or interference is non-stationary. Its main limitation is estimating the optimum rotational parameter 'a'. Current techniques choose 'a' that gives the minimum mean-square error (MMSE) between an SOI and its estimate. Such techniques are computational, and they do not provide good estimates when signal-to-noise ratio (SNR) or sample support is kept low, as is required in nonstationary environments. In this paper, we propose to estimate 'a' using Fractional Fourier domain decomposition (FFDD). We project the interference onto the FFDD basis vectors and choose 'a' that maximizes the projection. We show by simulation, using a non-stationary chirp channel function, that we estimate 'a' better than MMSE methods with just N = 4 samples down to E_b/N_0 = 3 dB. Averaging over M = 10 trials improves accuracy to E_b/N_0 = 0 dB.**

***Keywords*—Fractional Fourier Transform, Domain Decomposition, Singular Valued Decomposition.**

I. INTRODUCTION

The Fractional Fourier Transform (FrFT) has a wide range of applications in fields such as optics, quantum mechanics, image processing, and communications. It is a very useful method for separating a signal-of-interest (SOI) from interference and/or noise when the statistics of either are nonstationary [6]. The FrFT enables us to translate the received signal to an axis in the time-frequency plane where the SOI and interference may be separable [1], when they are not separable in the frequency domain, as produced by the conventional Fast Fourier transform (FFT), or in the time domain. The FrFT of a function f(x) of order a is defined as [6]

$$\mathbf{F}^a[f(x)] = \int_{-\infty}^{\infty} B_a(x,x')f(x')dx', \qquad (1)$$

Where the kernel is (x, x′) is defined as

$$B_a(x,x') = \frac{e^{i(\pi\hat{\phi}/4-\phi/2)}}{|sin\phi|^{1/2}} \times e^{i\pi(x^2cot\phi - 2xx'csc\phi + x'^2cot\phi)}, \qquad (2)$$

$\phi = a\pi/2$, and $\hat{\phi} = sgn[sin(\phi)]$. This applies to the range $0 < |\phi| < \pi$, or $0 < |a| < 2$. In discrete time, we can model the N × 1 FrFT of an N × 1 vector **x** as

$$\mathbf{X}_a = \mathbf{F}^a\mathbf{x}, \qquad (3)$$

Where $\mathbf{F}^a$ is an N × N matrix whose elements are given by ([3] and [6])

$$\mathbf{F}^a[m,n] = \sum_{k=0, k\neq(N-1+(N)_2)}^{N} u_k[m]e^{-j\frac{\pi}{2}ka}u_k[n], \qquad (4)$$

and where $u_k[m]$ and $u_k[n]$ are the eigenvectors of the matrix **S** defined by [3]

$$\mathbf{S} = \begin{bmatrix} C_0 & 1 & 0 & \dots & 1 \\ 1 & C_1 & 1 & \dots & 0 \\ 0 & 1 & C_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & C_{N-1} \end{bmatrix}, \qquad (5)$$

and

$$C_n = 2cos(\frac{2\pi}{N}n) - 4. \qquad (6)$$

Numerous methods are presented in the literature for implementing the FrFT efficiently (see for example [2] and [3]).

When applying the FrFT to perform interference suppression, we must first estimate the rotational parameter 'a'. Conventional methods rely on choosing the value of 'a', 0 ≤ a ≤ 2, which produces the MMSE between a desired (training) signal and its estimate. Of course, when the environment is non-stationary, it is necessary to perform this estimation with very few samples, i.e. before the statistics of the received signal change. When this is not done, large estimation errors, which result in poor interference suppression, can occur. MMSE-based algorithms, however, are known to require a large number of samples in practice [7]; hence, their performance will be suboptimal in non-stationary environments.

International Journal of Recent Development in Engineering and Technology
Website: www.ijrdet.com (ISSN 2347 - 6435 (Online)) Volume 4, Issue 9, September 2015)

Here, we propose to apply a technique based on the Fractional Fourier Domain Decomposition (FFDD) proposed in [9]. The technique is based on modeling the interference and noise environments using a decomposition similar to singular valued decomposition (SVD) except that the columns of the N × N matrix $\mathbf{F}^a$ are used in place of eigenvectors. The value of a_k for which the eigenvalue Λ_k is the largest is chosen to be the optimum value.

An outline of the paper is as follows: Section II describes the FFDD presented in [9]. Section III describes the proposed method for estimating the optimum value of 'a' using an approach based upon the FFDD. Section IV has simulation results showing the improvement of the proposed method over MMSE-based methods. Finally, conclusions and remarks on future work are given in Section V.

II. Background: Fractional Fourier Domain Decomposition (FFDD)

We follow the discussion presented in [9], but we assume, without loss of generality, that we have an N ×N matrix $\mathbf{H}_N$ which is the discrete time model of a non-stationary channel whose elements may be real or complex. Note that $\mathbf{H}_N$ could also be the kernel of a one dimensional interfering signal [9]. Let **h** represent a complex, time-varyingL×1 channel vector, written as

$$\mathbf{h} = [h_1\ h_2\ ...\ h_L]^T, \tag{7}$$

So that in matrix form, we can write $\mathbf{H}_N$ as

$$\mathbf{H}_N = \begin{bmatrix} h_1 & 0 & \dots & \dots & 0 \\ h_2 & h_1 & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots \\ h_L & h_{L-1} & \dots & \dots & \vdots \\ 0 & h_L & \dots & \dots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & h_L & h_{L-1} \\ 0 & 0 & \dots & 0 & h_L \end{bmatrix}. \tag{8}$$

For reference, we can write the singular value decomposition (SVD) of $\mathbf{H}_N$ as [9]

$$\mathbf{H}_N = \mathbf{U}_N \Sigma_N \mathbf{V}_N^H, \tag{9}$$

Where $\mathbf{U}_N$ and $\mathbf{V}_N$ are unitary N × N matrices whose columns are the eigenvectors of $\mathbf{H}_N\mathbf{H}^H_N$, and Σ_N is an N×N diagonal matrix whose elements are the positive square roots of the eigenvalues of $\mathbf{H}_N\mathbf{H}^H_N$.

Here, $(\cdot)^H$ denotes Hermitian (i.e. complex conjugate) transpose of the matrix $(\cdot)$.

The Fractional Fourier Domain Decomposition (FFDD) of $\mathbf{H}_N$, presented in [9], uses the FrFT matrix $\mathbf{F}^a$ to perform the decomposition. This is defined as [9]

$$\mathbf{H}_N = \sum_{k=1}^{K} \mathbf{F}^{-a_k} \mathbf{\Lambda}_k (\mathbf{F}^{-a_k})^H, \tag{10}$$

Where the Λ_k's are matrices whose diagonal elements contain the weighting coefficients similar to the weights contained in Σ_N in Eq. (9). Here, $a_{k=1} = 0$, $a_{k=K} = 2$, and we step k from 1,2,...,K, which results in stepping a_k from 0 to 2 using an appropriate step size. The step size will typically be between 0.01 and 0.1 and will determine the value of K.

Expanding Eq. (10), we can also write

$$\mathbf{H}_N = \sum_{k=1}^{K}\sum_{j=1}^{N} c_{kj}\mathbf{P}_{kj}, \tag{11}$$

Where c_{kj} is the j^{th} diagonal element of Λ_k. Also,

$$\mathbf{P}_{kj} = [\mathbf{F}^{-a_k}]_j([\mathbf{F}^{-a_k}]_j)^H, \tag{12}$$

Where $[F^{-a}{}_k]_j$ denotes the jth column of the N × N matrix $[F^{-a}{}_k]$.

III. Proposed Method For Estimating 'A' Using The FFDD

Note that the key difference between Eqs. (9) and (10) (or (11)) is that in Eq. (9) the basis functions for $\mathbf{H}_N$ are the columns of $\mathbf{U}_N$ and $\mathbf{V}_N$ whereas in Eqs. (10) or (11) the basis functions for $\mathbf{H}_N$ are the columns of $F^{-a}{}_k$, which are also basis functions for the a_k^{th} FrFT domain [9]. Hence, in Eqs. (10) and (11) we are projecting $\mathbf{H}_N$ onto subspaces given by the a_k^{th} FrFT domain.

Figure 1 shows the Wigner distribution (WD) of a desired signal s(t) and a corrupting channel h(t) and the optimum FrFT axis t_{ak} where the channel can best be filtered out, corresponding to the axis where the projection of the Wigner distribution of h(t) is maximum. The Wigner distribution is a time-frequency representation of a signal, and may be viewed as a generalization of the Fourier Transform, which is solely the frequency representation. The WD of a signal h(t) can be written as

$$W_h(t,f) = \int_{-\infty}^{\infty} h(t+\tau/2)h^*(t-\tau/2)e^{-2\pi j\tau f}d\tau. \tag{13}$$

International Journal of Recent Development in Engineering and Technology
Website: www.ijrdet.com (ISSN 2347 - 6435 (Online)) Volume 4, Issue 9, September 2015)

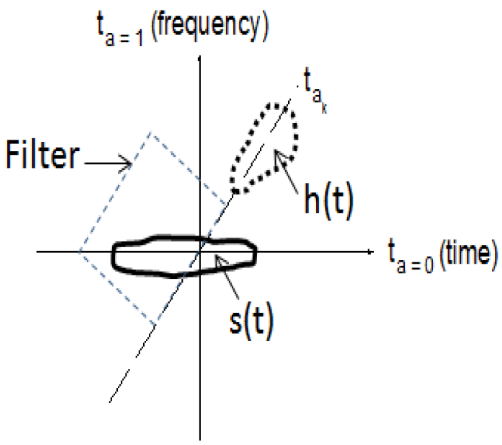

Fig. 1. Wigner Distribution of Signal s(t) and Channel h(t); optimum rotation axis t_{ak}

It is well-known that the projection of the WD of a signal h(t) onto an axis t_{ak} gives the energy in the FrFT, $|\mathbf{H}_{ak}(t)|^2$ (see e.g. [4] or [5]). Letting $\alpha_k = a_k\pi/2$, this is written as

$$|\mathbf{H}_{\alpha_k}(t)|^2 =$$

$$\int_{-\infty}^{\infty} W_h(tcos(\alpha_k) - fsin(\alpha_k), tsin(\alpha_k) + fcos(\alpha_k))df, \quad (14)$$

So this is the quantity to be maximized. Now, examining Eq. (10), we observe that the amount of energy contained in $\mathbf{H}_N$ in the domain given by a_k is given by the coefficient Λ_k. Based upon the above, we argue that by choosing the value of a_k where we obtain the maximum value of Λ_k in Eq. (10) gives us the FrFT domain in which the energy of the time-varying channel vector is maximum, so by rotating to this domain we can best filter it out. Thus, the problem is to determine the maximum Λ_k in Eq. (10). The solution is a maximization problem in which we compute

$$\underset{\substack{a_k \\ k=1,2,\ldots,K}}{arg\ max}\ \Lambda_k = max(diag([\mathbf{F}^{-a_k}(\mathbf{F}^{-a_k})^H]^{-1}\mathbf{H}_N)), \quad (15)$$

So that the value of a_k which produces the largest Λ_k is the optimum FrFT rotational parameter.

The solution in Eq. (15) can be traded for performance or complexity. If K is kept small, this means that the step size used to compute the a_k's is large, so we reduce complexity by computing fewer Λ_k's; however, this can result in lower estimation accuracy of the correct rotational value a_k and the rotational axis t_{ak}.

Alternately, if K is large, we obtain a better estimate of a_k by computing more values of Λ_k from Eq. (15) at the expense of more computations.

Note that the proposed solution requires estimation of the channel matrix $\mathbf{H}_N$ just as the MMSE-based method relies on a known training sequence [8]. Also, note that to compute the best 'a', the proposed method still requires a search over all values of 'a', and it still requires a matrix inversion similar to MMSE solutions, in this case inversion of the product of two FrFT matrices, as seen in Eq. (15). So, we do not expect there to be a computational complexity reduction with the proposed solution. However, because the proposed technique uses subspace projections, i.e. rank reduction similar to the SVD method, we expect that it will outperform the conventional MMSE methods, especially when the sample support is low. The proposed method also operates on the interference alone; therefore, we again expect improved performance over the MMSE method, which operates on both the SOI and interference. The performance improvement will be demonstrated by simulations in the next section.

IV. SIMULATIONS

We present simulation examples to illustrate the proposed method for calculating the optimum FrFT rotational parameter 'a' as summarized by Eq. (15). Without loss of generality, we let the signal-of-interest (SOI) be a digital binary sequence whose elements are in (−1,+1) that we would like to estimate in the presence of a non-stationary channel. We further assume the channel estimate is noisy, by corrupting it with additive white Gaussian noise (AWGN). Here, we ignore the carrier, and hence model the SOI as a baseband binary phase shift keying (BPSK) signal, denoted s(t). The number of bits per block is denoted N_1, and if we oversample each bit by a factor of SPB (samples per bit), the number of samples per block in the BPSK signal is $N = N_1SPB$.

We let the channel be modeled as time-varying, bandpass signal whose center frequency is changing with time, t, written as [4]

$$h(t) = e^{-j2\pi t^2} sinc(t). \quad (16)$$

This particular type of channel is chosen because it is a good example of a non-stationary channel [4]. We let the received signal be written as

$$r(t) = s(t) * h_n(t), \quad (17)$$

Where '∗' denotes convolution, and

$$h_n(t) = h(t) + n(t) \qquad (18)$$

is a noisy channel. The channel matrix $\mathbf{H}_N$ is computed from h(t) alone, without assuming any knowledge of n(t), so that we can include the effects of channel estimation errors. We let a_k be between 0 and 2 with a step size of 0.05. This implies that $a_1 = 0$, $a_2 = 0.05$, ..., $a_{41} = 2$ (so for this example K = 41). We adjust the noise variance to achieve a certain E_b/N_0 so that we can plot E_b/N_0 vs. the predicted optimum a_k, and we obtain the estimate by running M trials at each E_b/N_0 and computing the average over the M trials. We compare the predicted value of 'a' to that obtained using the MMSE method proposed in [8] in which we choose that value of 'a' for which the mean-square error between the true bit and its estimate is minimized.

Figures 2 to 5 show the results for M = 1, 10, 100, and 1000 trials, respectively. Note that the MMSE-based method fails at low E_b/N_0 for all values of M. The proposed technique continues to work down to E_b/N_0 = 0 dB until the number of trials reduces to M = 1, for which it then becomes subject to error, but only when $E_b/N_0 \leq 3$ dB. When M = 10 or more, the proposed method works over the range of E_b/N_0 from 0 to 20 dB. When the number of trials M is small but E_b/N_0 is high, the proposed technique produces more accurate estimates of the best 'a'. When E_b/N_0 and M are both large, i.e. greater than 8 dB and 100 respectively, both methods produce very accurate estimates, converging with very little error to the true value of a = 0.85. The true value is taken to be the value at which 'a' converges as M increases indefinitely and is also shown in the plots for reference.

For completeness, we modeled other non-stationary channels and saw similar performance improvements of the proposed FFDD method over the MMSE method. Two other channels that were studied are the Gaussian channel [8]

$$h(t) = \mathbf{A}e^{-\pi(t-\mathbf{s})^2}, \qquad (19)$$

Where **A** and **s** are uniformly distributed random variables, and a chirp channel given by [8]

$$h(t) = e^{j4\pi t^2}. \qquad (20)$$

We tested the proposed method using variations of h(t) given in Eq. (20), where time delays and frequency shifts were applied, with very similar results.

Note that we can also apply our technique to stationary channels, and it continued to outperform the MMSE method, but in that case conventional time (a = 0) or frequency (a = 1) based filtering methods could be used, and there is no benefit to the FrFT.

V. Conclusion

In this paper, we present a method for obtaining the best estimate of the rotational parameter 'a' when computing a Fractional Fourier Transform to suppress non-stationary interference, due for example to a time-varying channel, from a signal-of-interest (SOI). The approach is to perform a Fractional Fourier Domain Decomposition (FFDD) of the channel matrix using the columns of the FrFT matrix as the basis functions of the decomposition. Hence, the channel matrix is projected onto each rotational axis and we choose the one for which the projection is maximum, similar to a singular value decomposition. We compare the proposed method to an existing MMSE-based method and show that the proposed technique provides a much more robust estimate of 'a' at lower E_b/N_0, with fewer samples per trial, and with fewer trials. Future work includes expanding our prediction algorithm to study other forms of non-stationary interference, as well as a non-stationary SOI. We further seek to develop methods for predicting the optimum FrFT rotational parameter 'a' by analytical or numerical means without requiring a search over all possible values or requiring computationally expensive matrix inversions. We also seek to apply the newly developed algorithms to image processing applications.

Acknowledgments

The author thanks The Aerospace Corporation for funding this work and Alan Foonberg for reviewing the paper.

REFERENCES

[1] Almeida, L.B., "The Fractional Fourier Transform and Time-Frequency Representation", IEEE Trans. on Signal Processing, Vol. 42, No. 11, Nov. 1994.

[2] Candan, C., Kutay, M.A., and Ozaktas, H.M., "The Discrete Fractional Fourier Transform", Proc Int. Conf. on Acoustics, Speech, and Sig. Proc. (ICASSP), Phoenix, AZ, pp. 1713-1716, Mar. 15-19, 1999.

[3] Candan, C., Kutay, M.A., and Ozaktas, H.M., "The Discrete Fractional Fourier Transform", IEEE Trans. on Sig. Proc., Vol. 48, pp. 1329-1337, May 2000.

[4] Kutay, M.A., Ozaktas, H.M., Arikan, O., and Onural, L., "Optimal Filtering in Fractional Fourier Domains", IEEE Trans. on Sig. Proc., Vol. 45, No. 5, May 1997.

[5] Kutay, M.A., Ozaktas, H.M., Onural, L., and Arikan, O. "Optimal Filtering in Fractional Fourier Domains", Proc. IEEE International Conf. on Acoustics, Speech, and Signal Proc. (ICASSP), Vol. 2, pp. 937-940, 1995.

[6] Ozaktas, H.M., Zalevsky, Z., and Kutay, M.A., "The Fractional Fourier Transform with Applications in Optics and Signal Processing", John Wiley and Sons: West Sussex, England, 2001.

[7] Reed, I.S., Mallett, J.D., and Brennan, L.E., "Rapid Convergence Rate in Adaptive Arrays", IEEE Transactions on Aerospace and Electronic Systems, Vol. 10, pgs. 853863, Nov. 1974.

[8] Subramaniam, S., Ling, B.W., and Georgakis, A., "Filtering in Rotated Time-Frequency Domains with Unknown Noise Statistics", IEEE Trans. on Sig. Proc., Vol. 60, No. 1, Jan. 2012.

[9] Yetik, I.S., Kutay, M.A., Ozaktas, H., and Ozaktas, H.M., "Continuous and Discrete Fractional Fourier Domain Decomposition", IEEE, 2000.

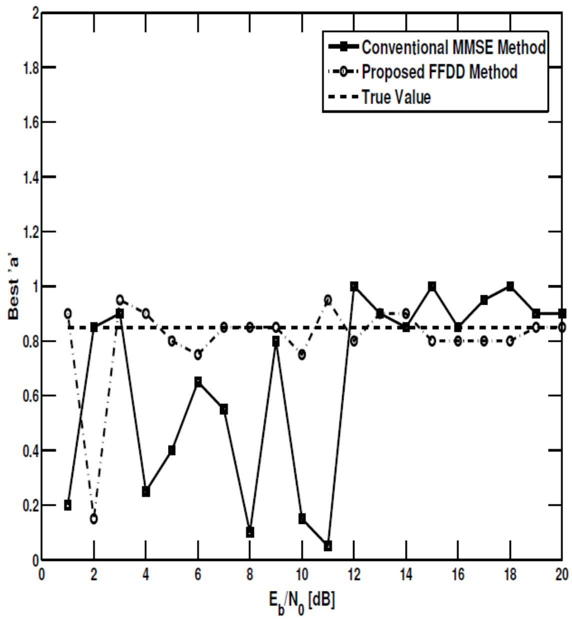

Fig. 2 E_b/N_0 [dB] vs. Best 'a'; M = 1 Trial

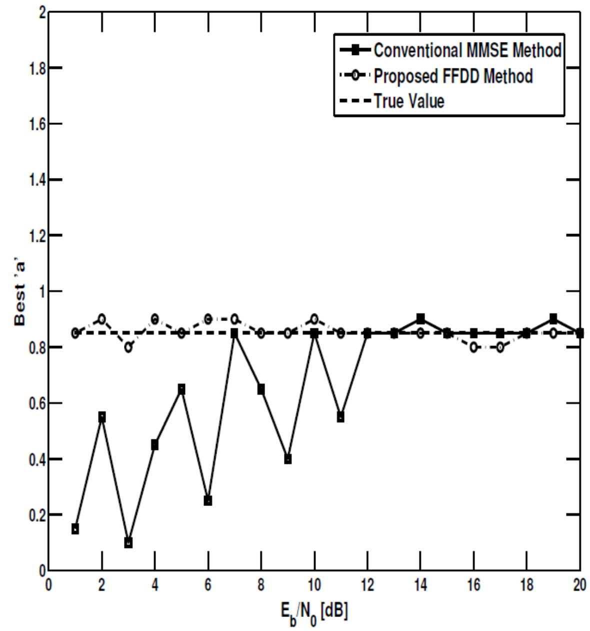

Fig. 3 E_b/N_0 [dB] vs. Best 'a'; M = 10 Trials

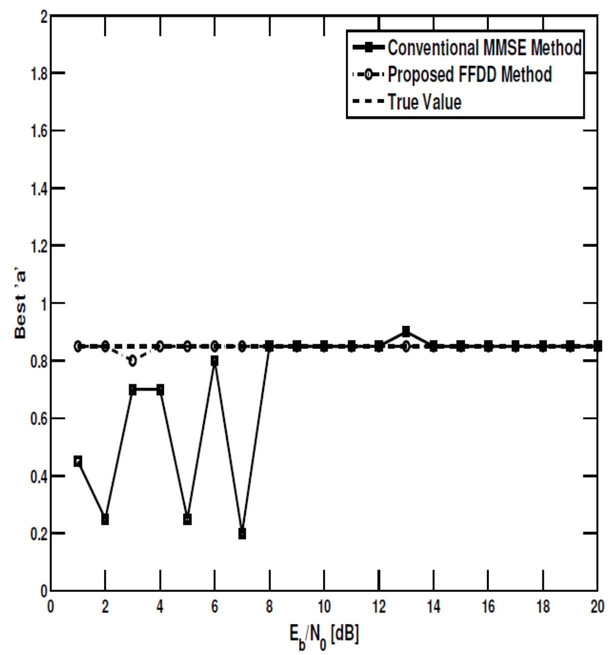

Fig. 4 E_b/N_0 [dB] vs. Best 'a'; M = 100 Trials

International Journal of Recent Development in Engineering and Technology
Website: www.ijrdet.com (ISSN 2347 - 6435 (Online)) Volume 4, Issue 9, September 2015)

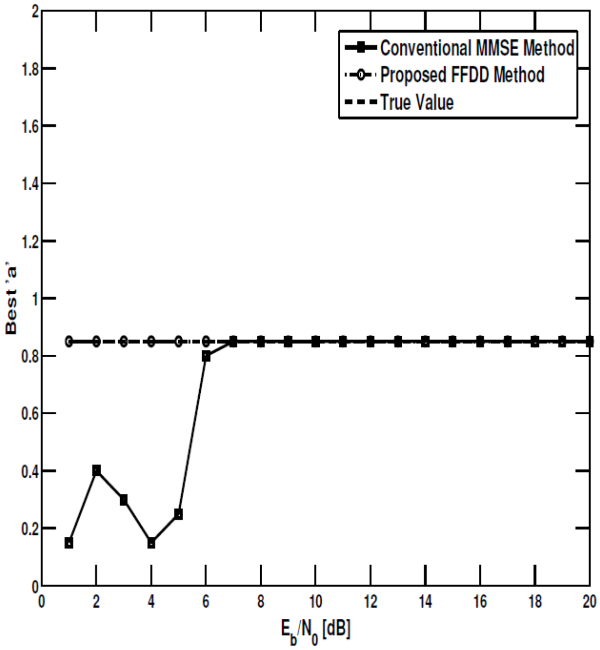

Fig. 5 E_b/N_0 [dB] vs. Best 'a'; M = 1000 Trials