International Journal of Recent Development in Engineering and Technology
Website: www.ijrdet.com (ISSN 2347-6435(Online) Volume 3, Issue 6, December 2014)

Geometric Continuity Two-Rational Cubic Spline with Tension Parameters

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***Abstract*— A smooth curve interpolation is very significant in computer graphics or in data visualization. In the present paper GC^1 -piecewise rational cubic spline function with tension parameter is considered which produces a monotonic interplant to a given monotonic data set. The parameters in the description of the spline curve can be used to modify the shape of the curve, locally and globally. It is observed that under certain condition the interpolant preserves the convexity property of data set. We have discussed the constraints for GC^2 rational cubic spline interpolant. The error analysis of the spline interpolant is also given.**

***Keywords*—Continuity, convexity, interpolation, monotonicity, rational spline.**

I. Introduction

Interpolation is a fundamental of scientific application, particularly when the data is achieving from the complex function or from scientific phenomena, it becomes crucial to incorporate the data in spline curve condition or in continuity.

Moreover, smoothness is an important term for pleasing visual display. Piecewise rational cubic spline functions provide powerful tools for designing curves, surfaces and some analytic primitives. Brodlie and Butt [1] and Delbourgo [2] have discussed the piecewise cubic interpolation of convex data Gregory and Delbourgo [5] consider the piecewise interpolation to monotonic data. Delbourgo and Gregory [3] have introduce tension parameters in the definition of the C^1-rational cubic spline function. The tension parameters have been so chosen that it provides the desired geometric shape to the rational interplant. For a designer these tension parameters act as intuitive tools for manipulating the shape of the curve. A C^2-piecewise rational (cubic/cubic) B'ezier curve involving two tension parameters which is used to interpolate the given monotonic data is described in [6]. The scheme for positive, monotone and convex data have developed by Sarfraz [8].

In the present paper a rational spline interpolant is constructed which matches given data values and at the same time preserves certain geometric features namely the monotonicity and convexity properties of the functions to be interpolated. In fact, we obtain a rational (cubic/linear) spline interpolant involving two tension parameters when value of the function and its first derivative are given at the knots.

Introducing two parameters α_i and β_i we construct GC^1-rational (cubic /linear)spline interpolant and obtain its error bounds in section 2. In section 3 we study the convexity and monotonicity of this rational spline interpolant. We have discussed the constraints for GC^2 rational spline interpolant in section 4.

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II. The Rational Spline Interpolation

Let P=$\{t_i\}_{i=1}^{n}$ where a = $t_1 < t_2 <$ ....$< t_n$ = b be a partition of the interval [a, b], let f_i , I = 1,2,….n be the function values at the data points.
We set

$$h_i = t_{i+1} - t_i \,,\ \Delta_i = (f_{i+1} - f_i)/h_i \qquad (2.1)$$

And

$$\theta = (t - t_i)/h_i \,, \qquad (2.2)$$

Further, we set

$$S(t) = P_i(\theta)/\ Q_i(\theta) \qquad (2.3)$$

Where

$$P_i(\theta) = \alpha_i f_i(1-\theta)^2 + [\lambda_i d_i h_i \alpha_i + (\beta_i + \alpha_i) f_i]\theta(1-\theta)^2 + [f_{i+1}(\beta_i + \alpha_i) - d_{i+1} h_i \beta_i]\theta^2(1-\theta) + \beta_i f_{i+1}\theta^2 \qquad (2.4)$$

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And

$$Q_i(\theta) = \alpha_i(1-\theta) + \beta_i\theta \qquad (2.5)$$

Here we chose parameter α_i , β_i in such a manner that

$$\alpha_i\,,\ \beta_i > 0 \quad \text{and} \quad \beta_i > \alpha_i \qquad (2.6)$$

This insures a strictly positive denominator in the rational spline.

S(t) satisfying the following interpolatory conditions;

$$S(t_i) = f_i \qquad S(t_{i+1}) = f_{i+1}$$

$$S'(t_i) = \lambda_i d_i \qquad S'(t_{i+1}) = d_{i+1} \qquad (2.7)$$

Where d'_i s denote the derivation values at knots t_i. For the error estimation of the piecewise rational cubic interpolating function defined by (2.3) , since the interpolation is local without loss of generality it is necessary only to consider the error in the subinterval $[t_i, t_{i+1}]$.

When f(t)$\epsilon C^2[a,b]$ and S(t) is the rational cubic interpolating function of f(t) in $[t_i, t_{i+1}]$. Consider the case that the knots are equally spaced namely $h_i = h$ for i = 1, 2,.. ..,n – 1 using the Peano-kernel theorem [5] gives the following

$$R[f]=f(t)-S(t)=\int_{t_i}^{t_{i+2}} f^2(\tau)R_t[(t-\tau)_+]d\tau,\ t\epsilon[t_i, t_{i+1}] \qquad (2.8)$$

$$R_t[(t-\tau)_+] = \begin{cases} p(\tau), & t_i < \tau < t \\ & = \\ q(\tau), & t_i < \tau < t_{i+1} \end{cases}$$

Where

$$p(\tau) = (t-\tau) - \frac{\theta^2[t_{i+1}-\tau]\{(1-\theta)(\alpha_i+\beta_i)+\beta_i\}-\theta^2(1-\theta)h_i\beta_i}{\alpha_i(1-\theta)+\theta\beta_i}$$

$$q(\tau)= \frac{-\{\theta^2[t_{i+1}-\tau]\{(1-\theta)(\alpha_i+\beta_i)+\beta_i\}-\theta^2(1-\theta)h_i\beta_i\}}{\alpha_i(1-\theta)+\theta\beta_i}$$

Then

$$\left|\left|R[f]\right|\right| = ||f(t)-S(t)||$$

$$\leq\left|\left|f^2(t)\right|\right|[\int_{t_i}^{t}|p(\tau)|\,d\tau]+\int_{t}^{t_{i+1}}|q(\tau)|d\tau + \int_{t_{i+1}}^{t_{i+2}}|r(\tau)|\,d\tau] \qquad (2.9)$$

q(τ) , since

$$q(\tau) = -\frac{\theta^2(1-\theta)h_i(\alpha_i+\beta_i)}{\alpha_i(1-\theta)+\theta\beta_i} \leq 0$$

and

$$q(t_{i+1}) = \frac{\theta^2(1-\theta)h_i\beta_i}{\alpha_i(1-\theta)+\theta\beta_i} \geq 0$$

It is easy to see that the root τ^* of q(τ) is

$$\tau^* = t_{i+1} - \frac{(1-\theta)h_i\beta_i}{(1-\theta)(\alpha_i+\beta_i)}$$

Thus

$$\int_t^{t_{i+1}}|q(\tau)d\tau = \int_t^{\tau^*}-q(\tau)d\tau + \int_{\tau^*}^{t_{i+1}}q(\tau)d\tau$$

$$= \frac{\theta^2(1-\theta)^2\,h_i{}^2[\left((1-\theta)(\alpha_i+\beta_i)\right)^2+\beta_i^2]}{2[(1-\theta)(\alpha_i+\beta_i)+\beta_i][\alpha_i(1-\theta)+\theta\beta_i]}$$

Similarly since

$$p(t) = q(t) \leq 0,\ p(t_i) = \frac{\theta(1-\theta)^2h_i\alpha_i}{\alpha_i(1-\theta)+\theta\beta_i} \geq 0$$

And the root τ_* of p(τ) in $[t_i, t]$ is

$$\tau_* = t_{i+1} - \frac{h_i(\alpha_i+\theta\beta_i)}{\alpha_i(1-\theta)+\theta\beta_i}$$

So that

$$\int_{t_i}^{t}|p(\tau)d\tau = \int_{t_i}^{\tau^*}p(\tau)d\tau + \int_{\tau^*}^{t_i}-p(\tau)d\tau$$

$$= \frac{\theta^2(1-\theta)^2\,h_i{}^2((\alpha_i+\beta_i)^2\theta^2+\alpha_i^2)}{2(\alpha_i(1-\theta)+\theta\beta_i)(\alpha_i(1+\theta)+\theta\beta_i)}$$

From the calculation above, it can be shown that

$$\left|\left|R[f]\right|\right| = ||f(t)-S(t)||$$

$$\leq\left|\left|f^{(2)}(t)\right|\right|h_i{}^2\,w\,(\theta,\alpha_i\beta_i)\ ,$$

Where

$$w\,(\theta,\alpha_i\beta_i) = w_1\,(\theta,\alpha_{i,}\beta_i) + w_2\,(\theta,\alpha_{i\,,}\beta_i)$$

$$w_1\,(\theta,\alpha_i\beta_i) = \frac{\theta^2(1-\theta)^2\,h_i{}^2((\alpha_i+\beta_i)^2\theta^2+\alpha_i^2)}{2(\alpha_i(1-\theta)+\theta\beta_i)(\alpha_i(1+\theta)+\theta\beta_i)}$$

$$w_2\,(\theta,\alpha_i\beta_i) = \frac{\theta^2(1-\theta)^2\,h_i{}^2(((1-\theta)\alpha_i+\beta_i))^2\theta^2+\alpha_i^2)}{2((1-\theta)(\alpha_i+\beta_i)+\beta_i)(\alpha_i(1-\theta)+\theta\beta_i)}$$

Based on the analysis above, there is the following.

Theorem 2.1. Let f(t)ϵGC^2[a, b] and S(t) be the rational cubic interpolating function of f(t) in $[t_i\,,t_{i+1}]$ defined by (2.3) for the parameters α_i, β_i

$$||R[f]|| = ||f(t)-S(t)|| \leq ||f^{(2)}(t)||\,h_i{}^2\,c_i\ ,$$

$$c_i = max_{0\leq\theta\leq1}\,w\,(\theta,\alpha_i,\beta_i)$$

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III. Convexity And Monotonicity Of The Interpolant

We shall now investigate monotonicity and convexity preserving properties of the rational (cubic/linear) interpolant to a given monotonic or convex data.

3.1. Monotonicity

Let f be a monotonic increasing function in [a, b] and $\lambda_i > 0$,

$$f_1 \le f_2 \le \ldots\ldots \le f_n \text{, or equivalently } \Delta_i \ge 0 \qquad (3.1)$$

We choose the derivative values d_i such that

$$d_i \ge 0\text{, i = 1, 2, …… n} \qquad (3.2)$$

We observe that s(t) is monotonic increasing if and only if for

$$t\epsilon[a, b],\ s^{(1)}(t) \ge 0 \qquad (3.3)$$

$$s^{(1)}(t) = \frac{\alpha_i{}^2\lambda_i d_i(1-\theta)^2+2\alpha_i\beta_i\Delta_i\theta(1-\theta)+\beta_i{}^2 d_{i+1}\theta^2+2(\lambda_i d_i\alpha_i+d_{i+1}\beta_i-(\alpha_i+\beta_i)\Delta_i)\theta(1-\theta)(\alpha_i(1-\theta)+\theta\beta_i)}{(\alpha_i(1-\theta)+\theta\beta_i)^2} \qquad (3.4)$$

We observe that the denominator of rational function $s^{(1)}(t)$ given in (3.4) is positive. Therefore considering in (3.4) we find that $s^{(1)}(t)$ is non-negative if

$$\frac{\alpha_i}{\beta_i} \ge \frac{(d_{i+1}-\Delta_i)}{(\Delta_i-\lambda_i d_i)} \qquad (3.5)$$

Therefore $s^{(1)}(t)$ is non-negative if (3.5) holds.

We have proved the following theorem.

Theorem 3.1**.**Given a monotonic increasing set of data satisfying (3.2), there exists a monotonic rational (cubic/linear) spline interpolant $s\epsilon GC'[a,b]$ involving the parameter α_i and β_i which satisfies the interpolatory conditions (2.7) provided (3.5) holds .

3.2. Convexity

Suppose the given data set is strictly convex then

$\Delta_1 < \Delta_2 < \ldots\ldots\ \Delta_{n-1}$.

We choose derivative values $d_i \ge 0$ to be such that

$$d_1 < \Delta_1 < d_2 < \Delta_{i-1} < d_i < \Delta_i < \ldots\ldots < d_n \qquad (3.6)$$

A simple calculation shows that for $t\epsilon[t_i, t_{i+1}]$, from (2.7) we get

$$s^{(2)}(t) = \frac{\{\frac{2}{h_i}[A_{2i}(1-\theta)^3+B_{2i}\big((1-\theta)^2+2\theta^2(1-\theta)\big)+C_{2i}(\theta(1-\theta)^2+\theta(1-\theta)+\theta^3)+D_{2i}\theta^3]\}}{[\alpha_i(1-\theta)+\theta\beta_i)]^3} \qquad (3.7)$$

Where

$$A_{2i} = \alpha_i^2[\alpha_i(\lambda_i d_i - \Delta_i) + \beta_i(d_{i+1} - \Delta_i)]$$

$$B_{2i} = \alpha_i^2\beta_i(\Delta_i - \lambda_i d_i)$$

$$C_{2i} = \alpha_i\beta_i^2(d_{i+1} - \Delta_i)$$

$$D_{2i} = \beta_i{}^2[\alpha_i(\Delta_i - \lambda_i d_i) + \beta_i(\Delta_i - \lambda_i d_{i+1})] \qquad (3.8)$$

We observe that $s^{(2)}(t)$ is non-negative if each of A_{2i}, B_{2i}, C_{2i} and D_{2i} is non-negative. Since we are assuming that (3.6) holds, C_{2i} is positive and B_{2i} is possitive if $\lambda_i < \frac{\Delta_i}{d_i}$. Thus the sufficient condition for the interpolant S(t) be convex is that $A_{2i} \ge 0, D_{2i} \ge 0$

Now $A_{2i} \ge 0$ if $\alpha_i^2[\alpha_i(\lambda_i d_i - \Delta_i) + \beta_i(d_{i+1} - \Delta_i)] \ge 0$

$$\text{i.e. } \frac{\alpha_i}{\beta_i} \ge \frac{(d_{i+1}-\Delta_i)}{(\Delta_i-\lambda_i d_i)}$$

$D_{2i} \ge 0$ if $\beta_i[\alpha_i(\Delta_i - \lambda_i d_i) + \beta_i(\Delta_i - \lambda_i d_{i+1})] \ge 0$

$$\text{i.e. } \frac{\alpha_i}{\beta_i} \ge \frac{(d_{i+1}-\Delta_i)}{(\Delta_i-\lambda_i d_i)}$$

There for the spline interpolant is convex if

$$\frac{\alpha_i}{\beta_i} \ge \frac{(d_{i+1}-\Delta_i)}{(\Delta_i-\lambda_i d_i)} \qquad (3.9)$$

Provided (2.6) and (3.6) holds.

Thus the spline interpolation is convex if (3.9) together with (2.7) and (3.6) holds. We have thus proved the following the theorem.

Theorem 3.2 For a given set of strictly convex data ,a convex rational (cubic/linear) spline interpolant $S\epsilon GC^1$[a, b] involving the parameters α_i and β_i exist which satisfies the interpolatory conditions (2.7), with the derivative parameters $d_i's$ satisfying (3.6) proved (2.6) and (3.9) holds.

IV. GC^2-Rational Spline Interpolant

For a given set of data points $\{(t_i, f_i)\}_{i=1}{}^n$ let s defined in 2, represent GC^2 rational (cubic/linear) spline interpolant.

For $[t_i, t_{i+1}]$, we have

$$s^{(2)}(t) = \frac{\{2[A_{2i}(1-\theta)^3+B_{2i}\big((1-\theta)^2+2\theta^2(1-\theta)\big)+C_{2i}\big(\theta(1-\theta)^2+\theta(1-\theta)+\theta^3\big)+D_{2i}\theta^3]\}}{h_i[\alpha_i(1-\theta)+\theta\beta_i)]^3}$$

Furthermore, this rational spline could even be GC^2 in the interpolating interval $[t_0, t_n]$

In fact, let

$$S''(t_i+) = \lambda_i^2 S''(t_i-)\ ;\ i = 1,2\ldots..\ n-1$$

The condition lead to the following continuous system of linear equations,

$$h_i\alpha_i\alpha_i\lambda_{i-1}\lambda_i^2 d_{i-1} + ((h_i\alpha_i\lambda_i^2(\beta_{i-1}-\alpha_{i-1}) + h_{i-1}\beta_{i-1}\lambda_i(\alpha_i-\beta_i)d_i + h_{i-1}\beta_i\beta_{i-1}d_{i+1} = h_i\alpha_i\beta_{i-1}\lambda_i^2\Delta_{i-1} + h_{i-1}\alpha_i\beta_{i-1}\Delta_i$$

Where the Δ_i' s, i=1,2…….n are given by (2.1)

Therefore, if the successive parameters $(\alpha_{i-1}, \beta_{i-1})$ and (α_i, β_i) satisfy (4.1) at i= 1,2 ….. n-1 , namely for the positive parameters $\alpha_{i-1}, \beta_{i-1}$ and the selected β_i, if

$$\alpha_i = \frac{h_{i-1}\beta_i\beta_{i-1}(d_{i+1}-\lambda_i d_i)}{h_i\left(\alpha_{i-1}\lambda_i^2(d_i-\lambda_i d_{i-1}) + \beta_{i-1}\lambda_i^2(\Delta_{i-1}-d_i)\right) + h_{i-1}\beta_{i-1}(\Delta_i-\lambda_i d_i)}$$

Then S(t) $\epsilon\ GC^2$[a, b].

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