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The Application of Augmented Lagrangian Methods in Traffic Equilibrium Problems

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***Abstract*—In this paper, we investigate tool setting as a policy to regulate the congestion of roads .According to example, the failure of congestion charging is illustrated. Instead, the strongly control is introduced to overcome this problem. By the advantages of augmented Lagrangian, a simple proof of the existence of optimal control for multiclass equilibrium problems where the value of time parameter varies continuously throughout the population is given.**

***Keywords*—network equilibrium problem, system optimal problem, strongly valid control, augmented Lagrangian methods.**

I. INTRODUCTION

In a transportation network subject to congestion, Congestion toll pricing addresses the classic traffic assignment problem for which Wardrop [1, 2] enunciated two principles of traffic flow: user-optimal behavioral hypothesis and the notion of system-optimality. The traditional objective of congestion pricing has been to determine link tolls which cause the solution of the tolled user-optimal problem to the optimal for the untolled system problem [3]. The system optimal refers to minimizing the total cost or the total time of network systems. Generally speaking, the system optimal network flows do not coincide with the equilibrium state. Therefore, from the perspective of network designers, they need to find a control that can make the user equilibrium meet the system optimal. The problem has attracted many researchers. In most of the literature, the one choice given has been the vector of marginal social cost pricing tolls. Although this approach is no longer valid when the valuation of travel varies across the population, it is yet true that the set of tolls that induces a system optimal use of the network is nonempty. In the case of finitely and infinitely many classes of customers, each characterized by its own value of time(VOT) parameter, [4] and [5] have shown that such tolls could be set to the optimal dual vector of network structure linear program, respectively.

But there no available literatures considering the strong effectiveness of congestion pricing, that is, after adding tolls to the network, whether all the equilibrium solutions achieve the system optimal.

This paper is structured as follows: in Section 2, we will give the traffic equilibrium model, and use an example to illustrate that only the congestion pricing alone can not make the all equilibrium states reach system optimum; in Section 3, we introduce the concept of strong valid control, and provide a simple proof of its existence.

II. THE MULTICLASS NETWORK EQUILIBRIUM MODEL

We begin this section with two problems:

System optimum problem

$$(SO) \qquad \min_X G(X) = < F(X), X > \quad s.t. X \in \Omega = \{X \geq 0 \mid AX = b\}$$

And the multiclass network equilibrium problem with control parameter

$$\text{VI(T)} \quad < \alpha F(X) + T, y - x > \geq 0, \forall y \in \Omega_\alpha \tag{1}$$

Where, the arc delay function $F: R^n \to R_+^n$ is positive, $\alpha \in [0, \alpha_{\max}]$ is the valuation of one unit of delay by the users, by which customer's time and cost can be integrated into the total cost. For example, the total cost of customer whose VOT is α on the arc i is $\alpha F_i(X) + T_i,\ i = 1, 2, \cdots, n$, X is the total flow, i.e. $X = \int_0^{\alpha_{\max}} x(\alpha) d\alpha$, $< \cdot, \cdot >_2$ is the product in R^n, $< \cdot, \cdot >$ is the product in $L^2(0, \alpha_{\max}; R^n)$, i.e.,

$$< f, g > = \int < f, g >_2, \forall f, g \in L^2(0, \alpha_{\max}; R^n).$$

For simplicity, we denote $< \cdot, \cdot >$ by the two kinds of inner product. Ω_α is the set of feasible arc flows:

$$\Omega_\alpha = \{x \in L^2(0, \alpha_{\max}; R^n) : Ax(\alpha) = bh(\alpha), x(\alpha) \geq 0, \forall \alpha \in [0, \alpha_{\max}]\},$$

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The set of corresponding total feasible arc flows Ω is defined:

$$\Omega = \{X : X = \int_0^{\alpha_{\max}} x(\alpha)d\alpha, x(\alpha) \in \Omega_\alpha\}. \qquad (2)$$

A is a $m \times n$ matrix, $h(\alpha)$ is measurable, and

$$h(\alpha) \geq 0, a.e., \int_0^{\alpha_{\max}} h(\alpha)d\alpha = 1.$$

In view of Wardrop's equilibrium principle, a feasible arc flow vector $x(\alpha)$ with T is the solution of the variational inequality VI(T).

Denote the solution sets of system optimum and multiclass network equilibrium problems by S^{so} and S_T, respectively. The purpose of the system designer [4, 5] is to find the valid arc tolls that can support an equilibrium flow as a system optimal flow. And they introduced the definition of valid tolls.

Definition 2.1 A tolls $T \in R_+^n$ is called a valid tolls if there exists a equilibrium flow x^{ue},such that the according total flow $X^{ue} \in S^{so}$.

By the strong duality of linear programming, [4] and [5] obtained the existence of the valid tolls. However, there is one issue needed to be addressed : most of these results appear to be not concerned with the inverse problem,i.e., for the given valid tolls, the user equilibrium may not be all the system optimal total flow. If the user equilibrium is not unique, the designer can not predict the effects of its tolls. Then the tolls may result in a highly total cost if an unexpected user equilibrium is realized.

For illustrative purpose, we present a simple example to show the problem.

Example 2.1 Consider a simple network consisting two nodes and two links. The links delay functions are given by

$$F_1(X) = 2, F_2(X) = \begin{cases} 1, & X_2 \in [0,1], \\ X_2, & X_2 \in [1,2], \end{cases}$$

With the demand $b = 2$. Then the system optimum problem:

Obviously, the unique solution of system optimum is

$$(SO) \qquad \min_X G(X) = \langle F(X), X \rangle$$
$$s.t. X \in \Omega = \{X_1, X_2 \geq 0 \mid X_1 + X_2 = 2\}.$$

Obviously, the unique solution of system optimum is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Now, let us check the user equilibrium problem with tolls

$$\left\langle \begin{pmatrix} 2+T_1 \\ F_2(X_2)+T_2 \end{pmatrix}, \begin{pmatrix} Y_1 - X_1 \\ Y_2 - X_2 \end{pmatrix} \right\rangle \geq 0, \forall X \in \Omega.$$

If $T = \langle T_1, T_2 \rangle$ is a valid toll, then in view of Wardrop's equilibrium condition,

$$2 + T_1 = F_2(X_2^s) + T_2. \qquad (3)$$

Therefore, $T_2 - T_1 = 1$. However, for any $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$, if $X_2 \in [0,1]$, and $X_1 + X_2 = 2$, then (3) holds, which implies that for any valid toll T, there exists a solution of equilibrium VI(T), such that its corresponding total flow does not belong to S^{so}.

In order to overcome this problem, we introduce a new definition

Definition 2.2 T is called a strongly control, if all the solutions x^{ue} of the equilibrium problem VI(T), we have $X^{ue} \in S^{so}$.

In next section, we will prove the existence of strongly control by the augmented Lagrangian method.

III. THE EXISTENCE OF STRONGLY VALID CONTROL

As the technique used in [5], we consider the following auxiliary linear programming with a given optimum X^s:

$$\begin{aligned} AP \quad & \min_x \quad \langle \alpha F(X^s, x) \rangle, \\ s.t. \quad & X - X^s \leq 0, \\ & x \in \Omega_\alpha, \end{aligned}$$

Set

$$\overline{X} = \int_0^{\alpha_{\max}} \alpha x(\alpha)d\alpha$$

$$\overline{\Omega} = \{(X, \overline{X}) : \exists x \in \Omega_x : X = \int_0^{\alpha_{\max}} x(\alpha)d\alpha, \overline{X} = \int_0^{\alpha_{\max}} \alpha x(\alpha)d\alpha\}.$$

Lemma 3.1 [5] Assume that the function $h(\alpha)$ is measurable over the interval $[0, \alpha_{\max}]$. Then the set $\overline{\Omega}$ is a polyhedron.

Based on above definitions, we can convert the above auxiliary infinite linear programming AP to the finite linear programming:

$$\begin{aligned} LP \quad & \min_{(X, \overline{X})} \quad \langle F(X^s, \overline{X}) \rangle, \\ s.t. \quad & X - X^s \leq 0, \\ & x \in \Omega. \end{aligned}$$

By the conservation of total flow, we have that following proposition.

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Proposition 3.1 The total flow X^* of the solution of problem AP is equal to X^s.

The augmented Lagrangian $L_c(X,\bar{X},c,p)$ of the optimization problem LP is defined as ([6]):

$$L_c(X,\bar{X},c,p) = < F(X^s), \bar{X} > + \sum_{i=1}^{n} r_i(X_i, X_i^s, c, p), \tag{4}$$

where

$$r_i(X_i, X_i^s, c, p) = \begin{cases} -\dfrac{p_i^2}{2c}, & X_i - X_i^s + \dfrac{p_i}{c} \le 0, \\ \dfrac{c}{2}(X_i - X_i^s)^2 + p_i(X_i - X_i^s), & X_i - X_i^s + \dfrac{p_i}{c} \ge 0. \end{cases}$$

Let P_c^* be the set of solutions of the dual problem:

$$\max_{p \ge 0} \psi_c(p),$$

Where the augmented Lagrangian dual function ψ is defined as

$$\psi_c(p) = \min_{(X,\bar{X}) \in \Omega} L_c(X,\bar{X},c,p).$$

It is well known that the augmented Lagrangian methods have many advantages over the general Lagrange mehtods [7]. Since the proof technique of the following theorem are similar to [7], we omit it.

Theorem 3.1 Consider the augmented Lagrangian function $L_c(X,\bar{X},p)$, defined by (4). Then

(i) L_c is difierentiable in $(X,\bar{X})$ and p;

(ii) L_c is stable, i.e., arg min $L_c(X,\bar{X},p^*) = \{(X^s,\bar{X}^s)\}, \forall p^* \in P^*$.

Based on the above theory, we give our main result.

Theorem 3.2 For any $p^* \in P^*$, there exists a T, , such that all the solutions of VI(T) belong to S^{so}:

$$\text{VI(T)} \qquad \langle \alpha F(X) + T, y - x \rangle \ge 0, \forall y \in \Omega_\alpha. \tag{5}$$

Proof: For any $p^* \in P^*$, consider the following programming problem:

$$\text{LP}_\text{C} \qquad \min_{(X,\bar{X}) \in \Omega} L_c(X,\bar{X},p^*).$$

Recalling Theorem 3.1, the unique solution of LP_c is (*Xs;X¹s*). Since $L_c(\cdot,\cdot,c,p^*)$ is convex and differentiable, then (*LPc*) is equivalent to the variational inequality:
(VI_c):

$$\left\langle \begin{pmatrix} \partial_X L_c(X^s,\bar{X}^s,c,p^*) \\ \partial_{\bar{X}} L_c(X^s,\bar{X}^s,c,p^*) \end{pmatrix}, \begin{pmatrix} X - X^s \\ \bar{X} - \bar{X}^s \end{pmatrix} \right\rangle \ge 0, \forall (X,\bar{X}) \in \bar{\Omega}.$$

i.e.,

$$\left\langle \begin{pmatrix} \partial_X L_c(X^s,\bar{X}^s,c,p^*) \\ F(X^s) \end{pmatrix}, \begin{pmatrix} X - X^s \\ \bar{X} - \bar{X}^s \end{pmatrix} \right\rangle \ge 0, \forall (X,\bar{X}) \in \bar{\Omega}. \tag{6}$$

which implies

$$\langle \alpha F(X) + \partial_X L_c(X^s,\bar{X}^s,c,p^*), x - x^s \rangle \ge 0.$$

where

$$\partial_X L_c(X,\bar{X},c,p^*) = \begin{pmatrix} \partial_{X_1} r_1(X_1, X_1^*, c, p_1^*) \\ \partial_{X_2} r_2(X_2, X_2^*, c, p_2^*) \\ \vdots \\ \partial_{X_n} r_n(X_n, X_n^*, c, p_n^*) \end{pmatrix},$$

and

$$\partial_{X_i} r_i(X_i, \bar{X}_i^*, c, p_i^{**}) = \begin{cases} 0, & X_i - X_i^* + \dfrac{p^*}{c} \le 0, \\ c(X_i - X_i^*) + p_i^*, & X_i - X_i^* + \dfrac{p^*}{c} \ge 0, \end{cases} \quad i = 1,2,\cdots,n.$$

Let $T = \partial_X L_c(X^s,\bar{X}^s,c,p^*)$. Then, VI(T) has a unique solution $(X^s,\bar{X}^s)$: Therefore, we complete our proof.

IV. Conclusion

In this paper, we have introduced and analyzed the notion of strongly valid control for the standard traffic assignment problem with fixed demand. By the stability of augmented Lagrangian, we investigate the existence of strongly valid controls for multiclass equilibrium problems.

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