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Modules Embedded in a Flat Module and their Approximations

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***Abstract*--An *F*- module is a module which is embedded in a flat module *F*. Modules embedded in a flat module demonstrate special generalized features. In some aspects these modules resemble torsion free modules. This concept generalizes the concept of modules over *IF* rings and modules over rings whose injective hulls are flat. This study deals with F-modules in a non-commutative scenario. A characterization of *F*-modules in terms of torsion free modules is also given. Some results regarding the dual concept of homomorphic images of flat modules are also obtained. Some possible applications of the theory developed above to *I* (*F*)-flat module approximation are also discussed.**

***Keyswords*-- 16E, 13Dxx**

I. INTRODUCTION

Approximations are an essential part all engineering computation and calculation. Through they are numerical in nature, in the background the algebraic formulation over whereas. This communication deals with this aspect of approximations.

For all homological concepts and notations [12] is referred. By an integral domain we mean a commutative integral domain and an *I*-ring *A* is a ring which is embedded in a ring *I*.

This study deals with *F*-modules that is, modules embedded in a flat module *F* [1], [2]. Here the definition of a flat module is adopted in the usual sense such as in [12]. In some aspects F modules resemble torsion free modules [2], [8], [9]. Colby has introduced the idea of *IF* rings [4]. In such rings every injective module is flat. Since an injective envelop exists for each module so in an *IF* ring every module is an F-module. Modules over rings in which injective hulls of modules are flat (*IHF*-ring) are studied by Khashyarmanesh and Salarian in [10]. F modules generalize modules over *IF* rings and *IHF*-ring modules whereas, F-rings generalize, *IF* rings and *IHF*-rings. In contrast to [5], [6], this study deals with F-modules in a non-commutative scenario. The dual idea of homomorphic images of flat modules is also discussed. Further, a characterization of *F*-modules in terms of torsion free modules is provided.

Finally, the theory developed above is applied to flat approximations particularly the *I* (*F*)-flat approximations.

We start with some necessary definitions.

1.1 Definition: $M \in$ mod-*A* is said to be a right ***F*-module** if it is embedded in a flat module $F \neq M \in$ mod-*A*. Similar definition holds for a left *F*-module.

1.2 Definition: A ring *A* is said to be an ***IF* ring** if its every injective module is flat [4].

1.3 Definition [Von Neumann]: A ring *A* is **regular** if, for each $a \in A$, and some another element $x \in A$ we have $axa = a$.

1.4 Examples: Every module in an *IF* ring is an *F*-module.
Every module over a regular ring is an *F*-module.
Over an integral domain, a module is torsionfree if and only if it is an *F*- module.

1.5 Definition: A ring *A* is said to be a right (respectively left) ***F*-ring** if every module in mod-*A* (respectively *A*-mod) is embedded in a flat module of mod-*A* (respectively of mod-*A*).

II. RING A AS F-MODULE:

Let *A* be an *I*-ring and *C* the quotient ring I/A then we have the exact sequence

$$0 \rightarrow A \rightarrow I \rightarrow C \rightarrow 0 \qquad (S)$$

In this regard we have the following proposition as in [2].

2.1 Proposition: Consider the sequence (S) in mod-*A* with the ring *I* to be flat in mod-*A*, then

(i) $Tor_n(C, M) = 0$ for each module *M* and all $n \geq 2$

(ii) The sequence $0 \longrightarrow Tor(C, M) \longrightarrow A \otimes M \longrightarrow I \otimes M \longrightarrow C \otimes M \longrightarrow 0$ is exact.

(iii) The sequence $0 \longrightarrow A \otimes M \longrightarrow I \otimes M \longrightarrow C \otimes M \longrightarrow 0$ is exact when *M* is *C*-torsion free.

2.2 Definition: A module M in A-mod is said to be ***C*-torsion free** if $Tor(C,M) = 0$. It is clear that a torsion free module is C-torsion free.

In the perspective of definition 2.2, we have the following corollary of proposition 2.1.

2.3 Corollary: If I is flat in mod-A, then M, C-torsion free in A-mod implies that the sequence $0 \longrightarrow M \longrightarrow I \otimes M \longrightarrow C \otimes M \longrightarrow 0$ is exact.

Proof: Consider the exact sequence $\cdots \longrightarrow Tor_1^A(I,M) \longrightarrow Tor_1^A(C,M) \longrightarrow A \otimes M \longrightarrow I \otimes M \longrightarrow C \otimes M \longrightarrow 0$. The first term is zero since I is flat and the second term is zero since M is C-torsion free. Hence the sequence $0 \longrightarrow M \longrightarrow I \otimes M \longrightarrow C \otimes M \longrightarrow 0$ is exact with $A \otimes M \cong M$.

It should be noted that if $\{M_i\}_{i \in I}$ is a family of F-modules (each M_i respectively embedded in flat modules F_i), then the direct Sum $\coprod_{i \in I} M_i$ is also an F-module. A ring is right? coherent, if the direct product of flat modules is flat and similarly the direct product of F-module is F-module.

F-modules resemble pure modules, so we will reproduce some results regarding pure modules.

2.4 Definition: An exact sequence $0 \longrightarrow M \longrightarrow N \longrightarrow S \longrightarrow 0$ (α) is said to be the pure exact if the sequence $0 \longrightarrow M \otimes X \longrightarrow F \otimes X \longrightarrow S \otimes X \longrightarrow 0$ is exact.

An exact sequence is not necessarily pure. In case X is flat, the sequence (α) is pure exact.

We use the following lemma Lam [18] to prove our next theorem.

2.5 Lemma: N in mod-A is flat if and only if the sequence $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ in mod-A is pure exact.

For F-modules some particular exact sequences can behave as pure exact sequences.

2.6 Proposition: Let M be an F-module in mod-A then the sequence $0 \longrightarrow M \longrightarrow F \longrightarrow F/M \longrightarrow 0$ (S) is pure if and only if F/M is flat.

Proof: Let M be an F-module which is embedded in the flat module F, then we show that $0 \longrightarrow M \longrightarrow F \longrightarrow F/M \longrightarrow 0$ (S) is pure. The sequence (S) will be pure if for all modules X in A-mod, the sequence $0 \longrightarrow M \otimes X \longrightarrow F \otimes X \longrightarrow F/M \otimes X \longrightarrow 0$ is exact. then there is a sequence $\cdots \longrightarrow Tor_1^R(F/M, X) \longrightarrow M \otimes X \longrightarrow F \otimes X \longrightarrow F/M \otimes X \longrightarrow 0$. $Tor_1^R(F/M, X) = 0$ because F/M is flat. Hence the sequence $0 \longrightarrow M \longrightarrow F \longrightarrow F/M \longrightarrow 0$ is pure.

Conversely if the sequence $0 \longrightarrow M \longrightarrow F \longrightarrow F/M \longrightarrow 0$ is pure then we prove that F/M is flat. The exact sequence $0 \longrightarrow M \longrightarrow F \longrightarrow F/M \longrightarrow 0$ is pure so $0 \longrightarrow M \otimes X \longrightarrow F \otimes X \longrightarrow F/M \otimes X \longrightarrow 0$ is exact where $X \in A$-mod. Hence F/M is flat.

2.7 Definition: A module is called *h.i.f.* module, if it is the homomorphic image of a flat module. If M in mod-A is an F-module then we have the exact sequence $0 \longrightarrow M \longrightarrow F \longrightarrow F/M \longrightarrow 0$ where F is flat. Then F/M is an *h.i.f.* module.

2.8 Proposition: IF F in mod-A is flat and I is left ideal, then the map $F \otimes I \rightarrow FI$ given by $f \otimes i \mapsto fi$ is an isomorphism. For further use we record the following two results of [21] without proof.

2.9 Proposition: If F is a right A-module such that $0 \rightarrow F \otimes I \rightarrow F \otimes A$ is exact for every f.g. (finitely generated) left ideal I of A, then F is flat

2.10 Proposition: Let M be an F-module in mod-A with F flat and the sequence

$$\longrightarrow M \longrightarrow F \longrightarrow F/M = Q \longrightarrow 0 \qquad \text{(T)}$$

exact. Then the following conditions are equivalent:

(i) Q is flat.

(ii) $M \cap FI = MI$ for every left ideal I.

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(iii) $M \cap FI = MI$ for every f.g (finitely generated) left ideal *I*.

Using 3.2 (6) a characterization of can be given as follows.

2.11 Proposition: If *A* is an integral domain then $Q \in$ mod-*A* is *h.i.f.* if and only if it is isomorphic to the quotient of a flat module.

Proof: Since *A* is an integral domain, every torsion free module is flat and the result is obvious in view of proposition 3.2 [6]. Finally we have the folloing.

2.12 Proposition: Let *M* in mod-*A* be an *F*-module and satisfy condition (ii) or (iii) of proposition 3.4 then for $X \in$ *A*-mod the sequence $0 \longrightarrow M \otimes X \longrightarrow F \otimes X \longrightarrow F/M \otimes X \longrightarrow 0$ is exact.

Proof: The exact sequence $0 \longrightarrow M \longrightarrow F \longrightarrow F/M \longrightarrow 0$ yields the exact sequence $Tor_1^A(F/M, X) \longrightarrow M \otimes X \longrightarrow F \otimes X \longrightarrow F/M \otimes X \longrightarrow 0$ from which the result is clear.

III. I-Flat F-Approximations

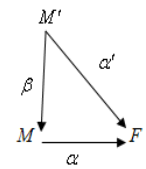
Now we introduce a modified definition of flat modules that of *I*-flat modules. We will apply some of the theory developed above to module approximations [8], [9], [10] and [11]. Our particular interest lies with flat approximations and more particularly with *I*-flat approximations and approximations by *I*(*F*)-flat modules.

3.1 Definition: Let *A* be an *I*-ring, *M* in *A*-mod is said to be ***I*(*n*)-cotorsion module** (*I*-cotorsion module of order *n*) if $Ext^n(I, M) = 0$ for all $n \geq 0$.

3.2 Definition: Let *A* be an *I*-ring, *N* in *A*-mod is said to be ***I* (*n*)-flat** (***I*-flat of order *n***) if $Ext_A^n(N, M) = 0$ for every *I*(*n*)-cotorsion module.

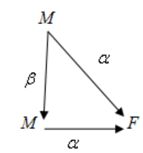
3.3 Definition: A module *M* in *A*-mod is said to be ***I* (*F*)-flat module** if it is embedded in an *I*-flat module ***F***.

3.4 Definition: Let *M* be an *I* (*F*)-flat module and $\alpha : M \rightarrow F$ (*F*, *M* $\in$ *A*-mod). Then α is called an ***I* (*F*)-flat precover** if for any other *I* (*F*)-flat *F*-module M' and a homomorphism $\alpha' : M' \rightarrow F$, the diagram



is commutative i. e. $\alpha \beta = \alpha'$.

3.5 Definition: If the diagram with M' replaced by *M* can only be completed to commutate when β is an automorphism. That is



$\alpha \beta = \alpha$, then α is said to ***I* (*F*)-flat cover**.

3.6 Lemma: Direct sum of *I* (*F*)-flat is *I* (*F*)-flat.

Proof: Proof: Let M_i be a family of *I* (*F*)-flat. Then for an *I*-flat module *F*, $Ext^n(F, M_i) = 0$ for each *i*, $Ext^n(\coprod F, M_i) \cong \prod Ext^n(F, M_i)$ implies the result.

For direct products it is not true.

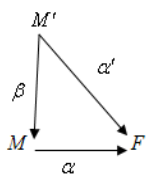
3.7 Proposition: $\prod F_i$ has an *I* (*F*)-flat cover if and only if each F_i does.

Proof: Let $\alpha_i : M_i \longrightarrow F_i$ be the flat covers for each i, then ($\coprod M_i \longrightarrow \coprod F_i \longrightarrow \prod F_i$) is an *I* (*F*) -flat precover (direct sum of *I*(*F*)-flat modules is *I* (*F*)-flat $Ext^n(\coprod_{i \in I} F_i, M) \cong \prod_{i \in I} Ext^n(F_i, M) = 0$ for *F* is *I*-flat module). Therefore $\prod F_i$ has *I*(*F*)-flat cover.

Conversely let $M \longrightarrow \prod F_i$ be an *I* (*F*)-flat cover then $M \longrightarrow \prod F_i \longrightarrow F_j$ is a *I*(*F*)-flat precover which is a cover.

3.8 Proposition: An I (F)-flat precover $\alpha : M \longrightarrow F$ is a cover if ker α contains non zero direct summand of M.

Proof: If $\alpha : M \longrightarrow F$ is a I (F)-flat cover and homomorphism $\alpha' : M' \longrightarrow F$ be a I (F)-flat precover then the following diagram



is commutative i. e. $\alpha\ \beta = \alpha'$. Then β is surjective and ker β is a direct summand of M'.

Finally, we have the following.

3.9 Proposition: If A is a coherent ring then the following conditions are equivalent:

(i) All injective modules in A-mod have I-flat covers.

(ii) All modules in A-mod have covers by $I(F)$ - flat module.

Proof: (2) implies (1): Let M be $I(F)$–flat module in A-mod which is embedded in I-flat module F and E is an injective module in A-mod. The homomorphism $\alpha : M \longrightarrow E$ is a cover of E. The morphisms extend $M \longrightarrow E$ to $F \longrightarrow E$ then the homomorphism $\alpha' : F \longrightarrow E$ is a precover of an injective module E.

(1) implies (2): Let X be a submodule of E in A-mod. If $\alpha : F \longrightarrow E$ is a flat cover and $M = \alpha^{-1}(X)$, then $M \longrightarrow X$ is a precover of X by a $I(F)$ – flat module.

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