

A Game–Theoretic Approach to Fuzzy Differential Equations

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Abstract-- In this work, we have tried to a good relationship among differential equation and Fuzzy–game along with Game theory.

Keywords-- Differential equation, player; Co-operative game, Pay-off, fuzzy game.

I. INTRODUCTION

Here, this paper introduces the multiple system of differential equations that characterize the prospective behavioural patterns of game players. These solutions instantiate a particular solution concept.

System of Differential Equations

R^n is Euclidean space and $\{x^i\}_{i=1}^m$ be a fixed set of unit vectors in R^n Here, $i = 1, 2, 3, \dots, m$.

We define the following functions:

$$g^i(x, b) = \langle a^i, x \rangle + b_i \in R^n;$$

In this, $b = \{b_1, b_2, b_3, \dots, b_m\} \in R^m$ and $\langle \cdot \rangle$ is denote the standard inner product on R^n .

Now, here we also define: $P^i(b) = \{x: g^i(x, b) = 0\}$ and elemental $(b)^v = \{x: g^i(x, b) \leq 0\}$

Clearly, the above $P^i(b)$ is a hyper plane in R^n . Let us examine the set of differential equations below:

$$\frac{dx}{dt} = x = D(x, b, K) = \sum_{i=1}^m K_i [g^i(x, b)] a^i \quad \text{--- (1)}$$

Wherever, $[\cdot]^+ = \max \{ \cdot, 0 \}$.

Evidently, the above function $D(x, b, K)$ follows directly that the preceding expression is continuous. Hence the following result follows:

Result–1

For any $b \in R^m, K \in R_+^m, x^0 \in R^n$, there is one specific solution, and no other.

$r(t, x_0, b, K)$ to (1), will be uninterrupted for $t \in (-\infty, \infty)$ and to the extent that $r(0, x_0, b, K) = x_0$.

1. Game with side application of differential equations to co-operative payments:

An “ n –person co-operative game with side payment” consists of:

1. Counting Numbers $N = \{1, 2, 3, \dots, n\}$ of n –players,

2. “Coalitions” of the players i.e., $P(N) - \phi = \left\{ \frac{S}{S} \subseteq N \right\}$,
3. “Characteristic function” denoted by $v: P(N) - \phi \rightarrow R$,
4. A certain “set of pay–offs” in R^n .

A game is represented by (N, v) , or simply v , and is defined by its set N of players.

A pay–off $x \in R^n$ implies a potential and realized distribution of some exchangeable merchandise among the given players wherever every player i accepts x_i .

To standardly quantify the level of contentment or discontent of definition along with a payoff indicated by matrix.

As the excess $e_S(x)$ decreases, coalition S becomes more satisfied with its payoff x .

At this juncture, let us also define the term "efficient excess" of S at x for $S \neq N, \emptyset$ to be:

$$\hat{e}_S(x) = \langle -A^S, x \rangle + \left(\frac{|N|V(S)}{|S|(|N|-|S|)} - \frac{V(N)}{|N|-|S|} \right) \quad \text{--- (2)}$$

Here $|S|$ is the size and order (cardinality) of S , $|N| = n, A^S \in R^n$.

Thus, In this, $A_i^S = \frac{1}{|S|}, i \in S$ and also $\frac{-1}{|N|-|S|}, i \in S$.

II. EXPLANATION

The two approaches to solution concepts:

Solution concepts generally fall into two main categories:

1. Fairness-Based concepts (Cooperative / Bargaining Games)
2. Stability-Based concepts (Non-cooperative Games)

1. Fairness-Based concepts:

These models look at how players should divide a reward or cost based on pre-defined principles of equity (axioms).

A. The Shapley Value: Distributes payoffs on the basis of every player’s marginal giving support of the group. It is uniquely defined by four axioms: Efficiency, Symmetry, Dummy player, and Additivity.

B. The Nash Bargaining Solution: A solution for two or more players that maximizes the product of their utility gains relative to a status quo.

It is bounded by axioms like Symmetry, scaling invariance, Pareto optimality, several equivalent terms and foundational properties.

2. Stability-based solutions:

These solutions do not assume fairness. Instead, they predict outcomes based on self-interest, where the game has reached a Nash equilibrium.

The “core” is the Pareto-optimal point which are rational for all subsets S . Explicitly, $Core(v) = \{x \in E(v) \mid e_S(x) \leq 0\}$ for all $S \in P(N) - \phi$

Now, we establish a streamlined negotiation framework for $\{A^S \in R^n, S \in P(N) - \phi\}$ and effective excess $\{e_S^A(x) \mid x \in R^n, S \in P(N) - \{\emptyset, N\}\}$ as follows:

We can model an efficient bargaining process as a system of simultaneous differential equations governing the evolution of player strategies.

where $K_S \in R$, for all $S \in P(N) - \{\phi, N\}$.

Here $P(N) - \{\phi, N\}$ is taken where the summation runs over the set of integers. The set $K_S > 0$, $S \in P(N) - \{\phi, N\}$ is called the set of the weight of a coalition, R_+^{2n-2} is obviously the set of like all this. The variable t may be taken to represent time. Plainly, system (2) of equation follows the exact same pattern as equation (1). Hence for any point x_0 , $\exists$ is continuous (in t) solution $r(t, x_0, v, K)$ whereby, $(x_0, v, K) = x_0$ along equation (2).

$$\frac{d}{dt} \sum_{i=1}^n rit(x, x_0, v, K) = 0$$

The following two lemmas follow:

Lemma-1.

If the initial point x_0 is efficient then $r(t, x_0, v, K)$ is efficient for all t .

With handling, we have:

Lemma-2.

For all set $S \neq N, \emptyset$, all $x \in E(v)$: $\frac{e_S(x)}{\|A^S\|^2} = e_x(x)$

It follows that:

$$\begin{aligned} Core(v) &= \{x \in E(v) \mid e_S(v) \leq 0, \forall S \neq N\} \\ &= \{x \in E(v) \mid \hat{e}_S \leq 0\} \quad \forall S \neq N \end{aligned}$$

Hence, we get the following:

Theorem

If initial point $x_0 \in E(v)$ then $r(e_S', x_0, v, K) = x_0$ is a solution of initial point.

III. FUZZY GAMES

A fuzzy game is defined as a mathematical framework in which player coalitions, strategic choices, or payoffs lack precision, utilizing fuzzy numbers instead of deterministic values. This game is resolved by suitable facts, quick conclusion, and get aim. It is clear that human thoughts and decisions are naturally complex and unpredictable, meaning anyone can make mistakes, correct them, and find that even a small amount of clarity greatly improves their search for knowledge. Consequently, achieving accurate for choices and goal may be unfeasible within a game. Thus, it becomes necessary to introduce fuzzy games.

Let $G = (N, v)$ be a crisp (non-fuzzy) n -person game of the set of natural number $N = \{1, 2, 3, \dots, n\}$, where the attribute function $v: P(N) \rightarrow R$ maps each union $S \subseteq N$ to a real value. Here, the notion $\vartheta(A)$ denotes the collective gain achieved by a coalition A , we can define the set τ^A as follows:

- $\tau^A(i) = 1$ if i is the element of A .
- $\tau^A(i) = 0$ if i is not the element of A .

An involvement rate $\tau^A(i)$ of a participant i denoted by $\tau^A(i) = 1$,

If competitor i takes part in A & $\tau^A(i) = 0$. if many a contestant does not take part in A .

So, coalition A is symbolized by $\tau^A = \{\tau^A(1), \tau^A(2), \dots, \tau^A(n)\}$.

Defined as partnership, τ^A fuzzy coalition allows a team member i to join with a varying participation cost $t_i \in [0, 1]$

In lieu of $\{0, 1\}$, A function whose outputs are real numbers represents the indicator function, or coalitional worth function, of a fuzzy game $f: [0, 1]^n \rightarrow R$. For each fuzzy coalition τ , let $f(\tau)$ be a real number. We denote this fuzzy game by the expression $FG = (N, f)$.

IV. CONCLUSION

By applying theory of fuzzy game, we can easily replicate the solution from the earlier portion, which may lead to even finer outcomes.

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