

Generalization of Comparative Growth Properties of Entire and Meromorphic Functions Related to the Relative (α, β, γ) -Order

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Abstract-- In this paper we study some results related to comparative growth properties of meromorphic functions on the basis of relative (α, β, γ) -order and relative (α, β, γ) -lower order of entire functions.

Keywords-- Entire function, Meromorphic function, Relative order, Nevanlinna characteristic function, Growth.

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I. INTRODUCTION, DEFINITION AND NOTATION

In this paper, we adopt the conventional notation and terminology of Nevanlinna value distribution theory associated with entire and meromorphic functions, as developed in the literature (see [1], [2], [3], [4], [5]). To avoid redundancy, the fundamental concepts are not recalled here.

Let $x \in [0, \infty)$ and let $k \in \mathbb{N}$, where $\mathbb{N}$ denotes the collection of all positive integers. The k -th iterates of the exponential and logarithmic functions are defined recursively by

$$\exp^{[k]}(x) = \exp(\exp^{[k-1]}(x)) \quad \text{and} \quad \log^{[k]}(x) = \log(\log^{[k-1]}(x)).$$

Throughout this work, we adopt the conventions

$$\log^{[0]}(x) = x, \quad \log^{[-1]}(x) = \exp(x), \\ \exp^{[0]}(x) = x, \quad \text{and} \quad \exp^{[-1]}(x) = \log(x).$$

For a meromorphic function f , the Nevanlinna characteristic function $T_f(r)$ is defined by

$$T_f(r) = m_f(r) + N_f(r),$$

where $m_f(r)$ denotes the proximity function of f , while $N_f(r)$ represents the integrated counting function of the poles of f in the disc $\{z: |z| \leq r\}$. Additional background on these functions may be found in [2]. The Nevanlinna characteristic function $T_f(r)$ for an entire function f is given by

$$T_f(r) = m_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta,$$

where $\log^+ x = \max\{\log x, 0\}$ for all $x \geq 0$.

Furthermore, if f is a non-constant entire function, then $T_f(r)$ is a strictly increasing and continuous function of r . Consequently, its inverse function $T_f^{-1}: (T_f(0), \infty) \rightarrow (0, \infty)$ exists and satisfies

$$\lim_{s \rightarrow \infty} T_f^{-1}(s) = \infty.$$

For the sake of completeness, we recall the following definition.

Definition 1.1. The order ρ_f and the lower order λ_f of a meromorphic function f are defined by

$$\rho_f = \limsup_{r \rightarrow \infty} \frac{\log T_f(r)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \rightarrow \infty} \frac{\log T_f(r)}{\log r}.$$

Next, let L denote a class of continuous non-negative functions $\alpha: (-\infty, \infty) \rightarrow [0, \infty)$ such that $\alpha(x) = \alpha(x_0) > 0$ for $x \leq x_0$ and $\alpha(x) \rightarrow \infty$ as $x \rightarrow \infty$. We say that $\alpha \in L_1$ if $\alpha \in L$ and

$$\alpha(a+b) \leq \alpha(a) + \alpha(b) + c,$$

for all $a, b \geq R_0$, where $c > 0$ is a constant.

Moreover, $\alpha \in L_2$ if $\alpha \in L$ and

$$\alpha(x + o(1)) = (1 + o(1))\alpha(x) \quad \text{as } x \rightarrow \infty.$$

Finally, α is said to belong to L_3 if $\alpha \in L$ and

$$\alpha(a+b) \leq \alpha(a) + \alpha(b), \quad \text{for all } a, b \geq R_0,$$

that is, α is a subadditive function. Clearly, $L_3 \subset L_1$. In particular, if $\alpha \in L_3$, then it is straightforward to verify that

$$\alpha(mr) \leq m\alpha(r), \quad m \geq 2,$$

where m is an integer. Up to a normalization, subadditivity may also be derived from concavity. Indeed, suppose that $\alpha(r)$ is concave on $[0, \infty)$ and satisfies $\alpha(0) \geq 0$. Then, for any $t \in [0, 1]$, we have

$$\alpha(tx) = \alpha(tx + (1-t) \cdot 0) \geq t \alpha(x) + (1-t)\alpha(0) \geq t \alpha(x).$$

Choosing $t = \frac{a}{a+b}$ or $t = \frac{b}{a+b}$, it follows that

$$\begin{aligned} \alpha(a+b) &= \frac{a}{a+b} \alpha(a+b) + \frac{b}{a+b} \alpha(a+b) \\ &\leq \alpha\left(\frac{a}{a+b}(a+b)\right) + \alpha\left(\frac{b}{a+b}(a+b)\right) = \alpha(a) + \alpha(b), \quad a, b \geq 0. \end{aligned}$$

Hence, concavity implies the subadditivity of α . Moreover, since $\alpha(r)$ is non-decreasing, subadditive, and unbounded, it satisfies

$$\alpha(r) \leq \alpha(r + R_0) \leq \alpha(r) + \alpha(R_0),$$

for any $R_0 > 0$. Consequently,

$$\alpha(r) \sim \alpha(r + R_0) \quad \text{as } r \rightarrow \infty.$$

Throughout this paper, we assume that $\alpha \in L_1$, $\beta \in L_2$ and $\gamma \in L_3$.

Heittokangas et al. [8] introduced the notion of φ -order for entire and meromorphic functions by considering φ as a subadditive function. Further developments along this direction were carried out by Belaïdi et al. [9], who generalized this concept and proposed the definitions of (α, β, γ) -order and (α, β, γ) -lower order of a meromorphic function. These notions are recalled below.

Definition 1.2. [9] The (α, β, γ) -order of a meromorphic function f , denoted by $\rho_{(\alpha, \beta, \gamma)}(f)$, and the (α, β, γ) -lower order, denoted by $\lambda_{(\alpha, \beta, \gamma)}(f)$, are defined as

$$\rho_{(\alpha, \beta, \gamma)}(f) = \limsup_{r \rightarrow \infty} \frac{\alpha(\log T_f(r))}{\beta(\log \gamma(r))},$$

and

$$\lambda_{(\alpha, \beta, \gamma)}(f) = \liminf_{r \rightarrow \infty} \frac{\alpha(\log T_f(r))}{\beta(\log \gamma(r))}$$

In general, the growth analysis of meromorphic functions has primarily been studied via Nevanlinna's characteristic function by comparison with exponential functions. However, when investigating the growth behavior of a meromorphic function relative to an entire function, the concept of relative growth indicators becomes relevant (see, for instance, [6],[7]).

To further advance the study of relative growth properties of meromorphic functions, Biswas et al. [10] introduced the notions of relative (α, β, γ) -order and relative (α, β, γ) -lower order with respect to an entire function. These concepts are defined as follows.

Definition 1.3. [10] Let f be a meromorphic function with respect to an entire function g . The relative (α, β, γ) -order of f with respect to g , denoted by $\rho_{(\alpha, \beta, \gamma)}(f)_g$, and the relative (α, β, γ) -lower order of f with respect to g , denoted by $\lambda_{(\alpha, \beta, \gamma)}(f)_g$, are defined as follows:

$$\begin{aligned} \rho_{(\alpha, \beta, \gamma)}(f)_g &= \limsup_{r \rightarrow +\infty} \frac{\alpha\left(\log^{[2]} T_g^{-1}\left(T_f(r)\right)\right)}{\beta\left(\log(\gamma(r))\right)} \\ \lambda_{(\alpha, \beta, \gamma)}(f)_g &= \liminf_{r \rightarrow +\infty} \frac{\alpha\left(\log^{[2]} T_g^{-1}\left(T_f(r)\right)\right)}{\beta\left(\log(\gamma(r))\right)} \end{aligned}$$

Remark 1.4. If $g(z) = z$, then Definition 1.3 reduces to Definition 1.2. Moreover, if $\alpha(r) = \beta(r) = \gamma(r) = r$ and $g(z) = z$, then Definition 1.3 coincides with the classical Definition 1.1.

Remark 1.5. An entire function f is said to have *regular relative (α, β, γ) -order* with respect to an entire function g if

$$\rho_{(\alpha, \beta, \gamma)}(f)_g = \lambda_{(\alpha, \beta, \gamma)}(f)_g.$$

In this paper, we study some results regarding the relative (α, β, γ) -order and the relative (α, β, γ) -lower order of meromorphic functions with respect to entire functions under a set of modified assumptions. Throughout the paper, it is assumed that all growth indicators involved are finite and non-zero.

II. LEMMAS

Lemma 2.1. Let g_1 and g_2 be two entire functions such that $T_{g_1}(r) \sim T_{g_2}(r)$. If f is meromorphic, then

$$\rho_{(\alpha,\beta,\gamma)}(f)_{g_1} = \rho_{(\alpha,\beta,\gamma)}(f)_{g_2}$$

Lemma 2.2. [13] Let f be a meromorphic function and g be an entire function, both of regular growth. Then

$$\rho_{(\alpha,\beta,\gamma)}(f)_g = \frac{\rho_{(\alpha,\beta,\gamma)}(f)}{\rho_{(\alpha,\beta,\gamma)}(g)}$$

III. MAIN RESULTS

Theorem 3.1. Let f be meromorphic and g, h be two entire functions with non-zero finite orders. Then

$$\liminf_{r \rightarrow \infty} \frac{\alpha \left(\log^{[2]} T_g^{-1} \left(T_f(r) \right) \right)}{\alpha \left(\log^{[2]} T_h^{-1} \left(T_f(r) \right) \right)} \leq \frac{\rho_{(\alpha,\beta,\gamma)}(f)_g}{\rho_{(\alpha,\beta,\gamma)}(f)_h} \leq \limsup_{r \rightarrow \infty} \frac{\alpha \left(\log^{[2]} T_g^{-1} \left(T_f(r) \right) \right)}{\alpha \left(\log^{[2]} T_h^{-1} \left(T_f(r) \right) \right)}$$

Proof. From the definition of the relative order, for any $\epsilon > 0$ and for all sufficiently large values of r , we have

$$\alpha \left(\log^{[2]} T_g^{-1} \left(T_f(r) \right) \right) < (\rho_{(\alpha,\beta,\gamma)}(f)_g + \epsilon) \beta \left(\log(\gamma(r)) \right), \quad (3.1)$$

while along a suitable sequence of values of r tending to infinity, it follows that

$$\alpha \left(\log^{[2]} T_g^{-1} \left(T_f(r) \right) \right) > (\rho_{(\alpha,\beta,\gamma)}(f)_g - \epsilon) \beta \left(\log(\gamma(r)) \right). \quad (3.2)$$

Similarly, for any $\epsilon > 0$ and all sufficiently large r , we have

$$\alpha \left(\log^{[2]} T_h^{-1} \left(T_f(r) \right) \right) < (\rho_{(\alpha,\beta,\gamma)}(f)_h + \epsilon) \beta \left(\log(\gamma(r)) \right), \quad (3.3)$$

and along a sequence of r tending to infinity,

$$\alpha \left(\log^{[2]} T_h^{-1} \left(T_f(r) \right) \right) > (\rho_{(\alpha,\beta,\gamma)}(f)_h - \epsilon) \beta \left(\log(\gamma(r)) \right). \quad (3.4)$$

Combining (3.1) and (3.4), we obtain along a sequence of r tending to infinity,

$$\frac{\alpha \left(\log^{[2]} T_g^{-1} \left(T_f(r) \right) \right)}{\alpha \left(\log^{[2]} T_h^{-1} \left(T_f(r) \right) \right)} < \frac{\rho_{(\alpha,\beta,\gamma)}(f)_g + \epsilon}{\rho_{(\alpha,\beta,\gamma)}(f)_h - \epsilon}$$

Since $\epsilon > 0$ is arbitrary, it follows that

$$\liminf_{r \rightarrow \infty} \frac{\alpha \left(\log^{[2]} T_g^{-1} \left(T_f(r) \right) \right)}{\alpha \left(\log^{[2]} T_h^{-1} \left(T_f(r) \right) \right)} \leq \frac{\rho_{(\alpha,\beta,\gamma)}(f)_g}{\rho_{(\alpha,\beta,\gamma)}(f)_h} \quad (3.5)$$

Similarly, using (3.2) and (3.3) along a sequence tending to infinity, we obtain

$$\frac{\alpha \left(\log^{[2]} T_g^{-1} \left(T_f(r) \right) \right)}{\alpha \left(\log^{[2]} T_h^{-1} \left(T_f(r) \right) \right)} > \frac{\rho_{(\alpha,\beta,\gamma)}(f)_g - \epsilon}{\rho_{(\alpha,\beta,\gamma)}(f)_h + \epsilon}$$

which implies, as $\epsilon > 0$ is arbitrary, that

$$\limsup_{r \rightarrow \infty} \frac{\alpha \left(\log^{[2]} T_g^{-1} (T_f(r)) \right)}{\alpha \left(\log^{[2]} T_h^{-1} (T_f(r)) \right)} \geq \frac{\rho_{(\alpha, \beta, \gamma)}(f)_g}{\rho_{(\alpha, \beta, \gamma)}(f)_h} \quad (3.6)$$

Hence, combining (3.5) and (3.6), we complete the proof. ■

Corollary 3.1. If g and h are of regular growth and $T_g(r) \sim T_h(r)$, then using Lemma 2.1 we get from Theorem 3.1 that

$$\liminf_{r \rightarrow \infty} \frac{\alpha \left(\log^{[2]} T_g^{-1} (T_f(r)) \right)}{\alpha \left(\log^{[2]} T_h^{-1} (T_f(r)) \right)} \leq 1 \leq \limsup_{r \rightarrow \infty} \frac{\alpha \left(\log^{[2]} T_g^{-1} (T_f(r)) \right)}{\alpha \left(\log^{[2]} T_h^{-1} (T_f(r)) \right)}$$

Theorem 3.2. Let f and h be two meromorphic and g be an entire function with non-zero finite order. Then

$$\liminf_{r \rightarrow \infty} \frac{\alpha \left(\log^{[2]} T_g^{-1} (T_f(r)) \right)}{\alpha \left(\log^{[2]} T_g^{-1} (T_h(r)) \right)} \leq \frac{\rho_{(\alpha, \beta, \gamma)}(f)_g}{\rho_{(\alpha, \beta, \gamma)}(h)_g} \leq \limsup_{r \rightarrow \infty} \frac{\alpha \left(\log^{[2]} T_g^{-1} (T_f(r)) \right)}{\alpha \left(\log^{[2]} T_g^{-1} (T_h(r)) \right)}$$

Proof. From the definition of the relative order, for any $\epsilon > 0$ and for all sufficiently large values of r , we have

$$\alpha \left(\log^{[2]} T_g^{-1} (T_h(r)) \right) < (\rho_{(\alpha, \beta, \gamma)}(h)_g + \epsilon) \beta \left(\log(\gamma(r)) \right), \quad (3.7)$$

while along a suitable sequence of values of r tending to infinity, it follows that

$$\alpha \left(\log^{[2]} T_g^{-1} (T_h(r)) \right) > (\rho_{(\alpha, \beta, \gamma)}(h)_g - \epsilon) \beta \left(\log(\gamma(r)) \right), \quad (3.8)$$

Now from (3.1) and (3.8), we obtain along a sequence of r tending to infinity that

$$\frac{\alpha \left(\log^{[2]} T_g^{-1} (T_f(r)) \right)}{\alpha \left(\log^{[2]} T_g^{-1} (T_h(r)) \right)} < \frac{\rho_{(\alpha, \beta, \gamma)}(f)_g + \epsilon}{\rho_{(\alpha, \beta, \gamma)}(h)_g - \epsilon}$$

Since $\epsilon > 0$ is arbitrary, it follows that

$$\liminf_{r \rightarrow \infty} \frac{\alpha \left(\log^{[2]} T_g^{-1} (T_f(r)) \right)}{\alpha \left(\log^{[2]} T_g^{-1} (T_h(r)) \right)} \leq \frac{\rho_{(\alpha, \beta, \gamma)}(f)_g}{\rho_{(\alpha, \beta, \gamma)}(h)_g} \quad (3.9)$$

Similarly, using (3.2) and (3.7) along a sequence of r tending to infinity, we obtain

$$\frac{\alpha \left(\log^{[2]} T_g^{-1} (T_f(r)) \right)}{\alpha \left(\log^{[2]} T_g^{-1} (T_h(r)) \right)} > \frac{\rho_{(\alpha, \beta, \gamma)}(f)_g - \epsilon}{\rho_{(\alpha, \beta, \gamma)}(h)_g + \epsilon}$$

which implies, as $\epsilon > 0$ is arbitrary, that

$$\limsup_{r \rightarrow \infty} \frac{\alpha \left(\log^{[2]} T_g^{-1} (T_f(r)) \right)}{\alpha \left(\log^{[2]} T_g^{-1} (T_h(r)) \right)} \geq \frac{\rho_{(\alpha, \beta, \gamma)}(f)_g}{\rho_{(\alpha, \beta, \gamma)}(h)_g} \quad (3.10)$$

Hence, from (3.9) and (3.10), we complete the proof. ■

Corollary 3.2. If f and h be meromorphic functions and g be an entire function, all of regular growth, then using Lemma 2.2 we get from Theorem 3.2 that

$$\liminf_{r \rightarrow \infty} \frac{\alpha \left(\log^{[2]} T_g^{-1} (T_f(r)) \right)}{\alpha \left(\log^{[2]} T_g^{-1} (T_h(r)) \right)} \leq \frac{\rho_{(\alpha, \beta, \gamma)}(f)}{\rho_{(\alpha, \beta, \gamma)}(h)} \leq \limsup_{r \rightarrow \infty} \frac{\alpha \left(\log^{[2]} T_g^{-1} (T_f(r)) \right)}{\alpha \left(\log^{[2]} T_g^{-1} (T_h(r)) \right)}$$

Theorem 3.3. Let f and h be meromorphic and g and k be entire functions with non-zero finite orders. Then

$$\liminf_{r \rightarrow \infty} \frac{\alpha \left(\log^{[2]} T_g^{-1} (T_f(r)) \right)}{\alpha \left(\log^{[2]} T_k^{-1} (T_h(r)) \right)} \leq \frac{\rho_{(\alpha, \beta, \gamma)}(f)_g}{\rho_{(\alpha, \beta, \gamma)}(h)_k} \leq \limsup_{r \rightarrow \infty} \frac{\alpha \left(\log^{[2]} T_g^{-1} (T_f(r)) \right)}{\alpha \left(\log^{[2]} T_k^{-1} (T_h(r)) \right)}$$

Proof. From the definition of the relative order, for any $\epsilon > 0$ and for all sufficiently large values of r , we have

$$\alpha \left(\log^{[2]} T_k^{-1} (T_h(r)) \right) < (\rho_{(\alpha, \beta, \gamma)}(h)_k + \epsilon) \beta \left(\log(\gamma(r)) \right), \quad (3.11)$$

while along a suitable sequence of values of r tending to infinity, it follows that

$$\alpha \left(\log^{[2]} T_k^{-1} (T_h(r)) \right) > (\rho_{(\alpha, \beta, \gamma)}(h)_k - \epsilon) \beta \left(\log(\gamma(r)) \right), \quad (3.12)$$

Now from (3.1) and (3.12), we obtain along a sequence of r tending to infinity that

$$\frac{\alpha \left(\log^{[2]} T_g^{-1} (T_f(r)) \right)}{\alpha \left(\log^{[2]} T_k^{-1} (T_h(r)) \right)} < \frac{\rho_{(\alpha, \beta, \gamma)}(f)_g + \epsilon}{\rho_{(\alpha, \beta, \gamma)}(h)_k - \epsilon}$$

Since $\epsilon > 0$ is arbitrary, it follows that

$$\liminf_{r \rightarrow \infty} \frac{\alpha \left(\log^{[2]} T_g^{-1} (T_f(r)) \right)}{\alpha \left(\log^{[2]} T_k^{-1} (T_h(r)) \right)} \leq \frac{\rho_{(\alpha, \beta, \gamma)}(f)_g}{\rho_{(\alpha, \beta, \gamma)}(h)_k} \quad (3.13)$$

Now from (3.2) and (3.11) along a sequence of r tending to infinity, we obtain

$$\frac{\alpha \left(\log^{[2]} T_g^{-1} (T_f(r)) \right)}{\alpha \left(\log^{[2]} T_k^{-1} (T_h(r)) \right)} > \frac{\rho_{(\alpha, \beta, \gamma)}(f)_g - \epsilon}{\rho_{(\alpha, \beta, \gamma)}(h)_k + \epsilon}$$

which implies, as $\epsilon > 0$ is arbitrary, that

$$\limsup_{r \rightarrow \infty} \frac{\alpha \left(\log^{[2]} T_g^{-1} (T_f(r)) \right)}{\alpha \left(\log^{[2]} T_k^{-1} (T_h(r)) \right)} \geq \frac{\rho_{(\alpha, \beta, \gamma)}(f)_g}{\rho_{(\alpha, \beta, \gamma)}(h)_k} \quad (3.14)$$

Hence, from (3.13) and (3.14), we complete the proof. ■

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