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Study of Group Theory Homomorphism Abstract Algebra

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Abstract-- **Group theory is one of the fundamental areas of abstract algebra. It provides a mathematical framework for studying symmetry, transformations, and algebraic operations. A central concept in group theory is the homomorphism, which is a structure-preserving mapping between groups. Homomorphisms allow mathematicians to compare groups, identify their common structural properties, and construct new groups from existing ones. This paper presents an introductory study of groups, subgroups, normal subgroups, quotient groups, group homeomorphisms, kernels, images, isomorphism, and the First Isomorphism Theorem. Examples are included to illustrate the role of homomorphism in understanding the structure of groups.**

Keywords-- **Abstract algebra, group theory, group, subgroup, homomorphism, kernel, image, isomorphism, quotient group.**

I. INTRODUCTION

Abstract algebra is the branch of mathematics concerned with algebraic structures and the relationships between them. Unlike elementary algebra, which primarily deals with numerical calculations, abstract algebra studies structures such as groups, rings, fields, and modules. Among these structures, groups occupy a particularly important position. The theory of groups studies algebraic operations that occur in many mathematical settings, including multiplication of numbers, addition of vectors, and composition of transformations. One of the most important ideas in group theory is that two apparently different groups may have essentially the same algebraic structure**.** Homomorphisms provide a systematic way of studying this relationship. A group homomorphism preserves the group operation and consequently provides information about the structure of the groups involved.

Group Theory Definition:

A group is a set G together with a binary operation $_$ that satisfies the axioms: For all $a, b, c \in G$:

1. *Closure:* $a _ b \in G$ Combining any two elements gives another element in the same set.

2. *Associativity:* $(a _ b) _ c = a _ (b _ c) $ the way you group elements doesn't matter.

3. *Identity Element:* There exists $e \in G$ such that $e _ a = a _ e = a$ There is an element that leaves others unchanged when combined.

4. *Inverse Element:* For each $a \in G$, there exists $a^{-1} \in G$ such that

$a _ a^ {-1} = a^{-1} _ a = e$

Every element has an "opposite" that brings you back to identity.

We write it as: $(G, *)$ is a group.

Simple Group

Theory is the branch of mathematics that studies algebraic structures called "groups". It studies symmetry, transformations, and how elements combine with each other under specific rules.

Think of it as: "The mathematics of symmetry and structure"

Basic Examples

Set Operation Is it a Group Integers $\mathbb {Z}$ Addition $+$ Yes0 is identity, $-a$ is inverse Positive Real Numbers Multiplication $\times$ Yes 1 is identity, $1/a$ is inverse

2×2 Matrices Matrix Multiplication Yes Used in physics for rotations on-Example: Natural numbers $\mathbb {N}$ under addition is not a group because there is no i A subset H of a group G is called a subgroup if H itself is a group under the same operation as G.

We write: $H \leq G$nverse. Ex: inverse of 3 is -3 which is not in $\mathbb {N}$.

Subgroup Test

$H \leq G$ iff:

1. $H \neq \emptyset$

2. For all $a, b \in H$, $a * b^{-1} \in H$

Example:

$G = (\mathbb{Z}, +)$ integers under addition

$H = (2\mathbb {Z}, +)$ = set of even integers {...-4, -2, 0, 2, 4...}

H is a subgroup because sum of two evens is even, inverse of even is even, and 0 is there.

Theorem: Lagrange's Theorem

If G is a finite group and $H \leq G$, then

|H| \text{ divides } |G|

Where $|G|$ = order of group = number of elements.

Let $G = \mathbb{Z}_6 = \{0,1,2,3,4,5\}$ under addition mod 6, so $|G|=6$

$H = \{0,2,4\}$ is a subgroup, $|H|=3$

A function $\phi: G \rightarrow G'$ between two groups is a homomorphism if:

\phi(a * b) = \phi(a) *' \phi(b) \quad \forall a,b \in G

It "preserves the group operation".

1. Monomorphism: 1-1 homomorphism
2. Epimorphism: Onto homomorphism
3. Isomorphism: 1-1 and onto homomorphism. We write $G \cong G'$
4. Endomorphism: $G \rightarrow G$
5. Automorphism: Isomorphism from G to itself

$\phi: (\mathbb{Z}, +) \rightarrow (\mathbb{Z}_2, +)$ defined by $\phi(n) = n \mod 2$

$\phi(3+5) = \phi(8) = 0$

$\phi(3) + \phi(5) = 1 + 1 = 0$

So $\phi(a+b) = \phi(a)+\phi(b)$. Hence homomorphism.

Solution: This maps all evens → 0, all odds →

Kernel and Image For homomorphism $\phi: G \right arrow G'$

Kernel: $Ker(\phi) = \{g \in G : \phi(g) = e'\}$ is a normal subgroup of G

Image: $Im(\phi) = \{\phi(g) : g \in G\}$ is a subgroup of G'

First Isomorphism Theorem

If $\phi: G \rightarrow G'$ is a homomorphism, then:

G / Ker(\phi) \cong Im(\phi)

"Quotient group is isomorphic to Image"

Example Solution:

Take $\phi: \mathbb{Z} \rightarrow \mathbb{Z}_6$, $\phi(n) = n \mod 6$

$Ker(\phi) = 6\mathbb{Z}$ = multiples of 6

$Im(\phi) = \mathbb{Z}_6$

By theorem: $\mathbb{Z} / 6\mathbb{Z} \cong \mathbb{Z}_6$ ✓

II. Types Of Groups In Group Theory

Groups are classified based on their properties like size, operation, and structure.

Based on Number of Elements Finite Group

A group with a finite number of elements.

Ex: D_3 = symmetries of triangle, $|D_3| = 6$

Ex: $\mathbb {Z}_6 = \{0,1,2,3,4,5\}$ under addition mod 6

Infinite Group:

A group with infinite number of elements.

Ex: $(\mathbb{Z}, +)$ = integers under addition

Ex: $(\mathbb{R}, +)$ = real numbers under addition Based on

Operation Property Abelian Group / Commutative Group

Where $a _ b = b _ a$ for all $a, b \in G$

Ex: $(\mathbb {Z}, +) $, $(\mathbb{R} - \{0\}, \times)$

Named after Niels Abel

Non-Abelian Group

Where $a _ b \neq b _ a$ for some a, b

Ex: $GL(n, \mathbb{R})$ = invertible matrices under multiplication

Ex: S_3 = permutations of 3 elements

Cyclic Group

A group that can be generated by a single element.

If $g \in G$, then $G = \{g^n : n \in \mathbb{Z}\}$

Ex: $\mathbb {Z}_4 = \{0,1,2,3\}$ under addition mod 4.

Generator = 1

Note: Every cyclic group is Abelian

Non-Cyclic Group

Cannot be generated by one element.

Ex: Klein-4 group $V_4 = \{e, a, b, c\}$

Special Types of Groups

Symmetric Group S_n

Group of all permutations of n elements.

Order = $n!$

Ex: S_3 has $3! = 6$ elements. This is the smallest non-*Abelian group.*

Dihedral Group D_n

Group of symmetries of a regular n-gon. Includes rotations + reflections.

Order = $2n$

Ex: D_4 = symmetries of square, $|D_4| = 8$

General Linear Group $GL(n, F)$

Group of all $n \times n$ invertible matrices over field F under matrix multiplication.

Quaternion Group Q_8

$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$

Non-Abelian group of order 8. Used in physics.

Trivial Group

Group with only one element: $\{e\}$

Subgroup

$H \leq G$ if H is also a group under same operation.

Ex: $2\mathbb{Z} \leq \mathbb{Z}$

Normal Subgroup

$N \trianglelefteq G$ if $g Ng^{-1} = N$ for all $g \in G$.

Used to form quotient groups G/N

Simple Group

A group with no normal subgroups except $\{e\}$ and itself.

"Building blocks" of all finite groups. Ex: $A _n$ for $n \geq 5$

Lagrange's Theorem

These two ideas are the foundation for understanding the structure and size of groups.

Let G be a group and $H \leq G$ be a subgroup.

For any $a \in G$:

1. Left Coset of H:

aH = \{a * h : h \in H\}

2. Right Coset of H:

Ha = \{h * a : h \in H\}

Think of a coset as a "shifted copy" of H inside G.

1. All cosets have the same size as H: $|aH| = |H|$
2. Any two left cosets are either identical or disjoint
3. G is the union of all distinct cosets of H
4. If G is Abelian, then $aH = Ha$

$G = \mathbb{Z}_6 = \{0,1,2,3,4,5\}$ under addition mod 6

$H = \{0, 2, 4\}$ is a subgroup

If G is a finite group and $H \leq G$ is a subgroup, then:

|H| \text{ divides } |G|

Also:

\text{Number of distinct cosets} = \frac{|G|}{|H|}

This number is called the index of H in G, written as $[G : H]$

Idea The distinct left cosets of H partition G into equal-sized disjoint sets.

Each coset has size $|H|$.

If there are k cosets, then $|G| = k \cdot |H|$ → So $|H|$ divides $|G|$.

$G = S_3$ = permutations of 3 objects, $|G| = 6$

$H = \{e, (12)\}$, $|H| = 2$

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By Lagrange: 2 divides 6 ✓

Number of cosets = $6/2 = 3$

Cosets: H, $(13)H$, $(23)H$

$G = \mathbb{Z}_{12}$, $|G| = 12$

Possible subgroup orders by Lagrange: 1, 2, 3, 4, 6, 12

Important Corollaries:

1. Order of an element divides order of group:

   If $a \in G$, then $|a|$ divides $|G|$

2. If $|G|$ is prime, then G is cyclic

   Because only subgroups possible are $\{e\}$ and G itself

3. Quick Problem with Solution

Q: Let $|G| = 15$. What are the possible orders of subgroups of G?

Solution:

By Lagrange's Theorem, $|H|$ must divide 15.

Divisors of 15 = 1, 3, 5, 15

So possible subgroup orders are: 1, 3, 5, 15

Q: In $\mathbb{Z}_{10}$, find the cosets of $H = \{0, 5\}$

Solution:

$0+H = \{0,5\}$

$1+H = \{1,6\}$

$2+H = \{2,7\}$

$3+H = \{3,8\}$

$4+H = \{4,9\}$

III. Applications And Importance

Homomorphisms are useful because they allow complicated algebraic structures to be studied through simpler ones.

They occur throughout modern mathematics. In group theory, Compare different groups; identify structural similarities; construct quotient groups; Study symmetry and transformations; establish isomorphism theorems; develop group representations. A group representation can itself be viewed as a homomorphism from a group into a group of invertible transformations. This connects abstract group theory with linear algebra the ideas of kernels, images, quotient structures, and homomorphisms also extend beyond groups to rings, modules, vector spaces, and other algebraic structures

IV. Conclusion

Group theory provides a fundamental framework for understanding algebraic structure and symmetry. Within group theory, homomorphisms are essential because they describe mappings that preserve the operation of a group.The kernel and image of a homomorphism provide important information about its behavior. The kernel is a normal subgroup of the domain, while the image is a subgroup of the codomain. The First Isomorphism Theorem connects these concepts by establishing the important relationship.

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