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Nabla Fractional Differential Calculus: A Projection-Based Framework for Discrete Fractional Dynamics

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Abstract-- **Fractional calculus extends classical differentiation and integration to arbitrary real or complex orders and provides a useful mathematical framework for systems exhibiting memory and hereditary effects. When such systems are represented by discrete observations, nabla fractional calculus offers a natural backward-oriented approach through fractional sums and differences. This paper presents the fundamental concepts of nabla fractional differential calculus and proposes a projection-based framework for representing historical information in discrete fractional systems. The proposed approach interprets the fractional state of a system as a weighted projection of its historical states through a fractional memory kernel. A matrix representation of the projection is also introduced, providing a possible computational framework for discrete fractional models. The proposed formulation may be useful in discrete dynamical systems, time-series analysis, signal processing, control systems, and numerical modelling.**

Keywords--**Nabla fractional calculus; fractional difference; discrete fractional calculus; backward difference; fractional sum; fractional memory; projection operator; memory kernel; discrete dynamical systems; fractional difference equations.**

I. Introduction

Fractional calculus is an extension of classical calculus in which differentiation and integration can be defined for non-integer orders. Unlike ordinary integer-order differentiation, fractional operators possess nonlocal characteristics and can therefore represent systems in which the present state depends upon previous states.

In many practical applications, observations are available at discrete time intervals. Examples include digital signals, financial data, biological observations, environmental measurements, and numerical simulations. For such applications, discrete fractional calculus provides an important mathematical framework.

The nabla operator is a backward difference operator defined by Repeated application produces higher-order differences, The development of discrete fractional calculus has extended these ideas to non-integer orders. Atici and Eloe established important properties of discrete fractional calculus using the nabla operator, including fractional sums, differences, transform methods, and fractional difference equations.

The purpose of this paper is not simply to reproduce existing definitions. Instead, it proposes a projection-based interpretation of nabla fractional calculus, in which the fractional state of a discrete system is considered a projection of its historical information through a fractional memory kernel.

II. Mathematical Preliminaries

Let be a discrete time domain. For a function, the first-order nabla difference is the higher-order nabla difference is defined recursively as for a fractional order, the nabla fractional sum can be represented using the rising factorial as where is the rising factorial. For a Riemann– Liouville-type nabla fractional difference may be represented schematically by The precise domain and starting-point conventions should be stated whenever a particular definition is used.

III. Fractional Memory

One of the important characteristics of fractional operators is their dependence on historical values. An ordinary difference, is primarily local. A fractional difference generally incorporates a larger portion of the available history. Thus, a fractional operator can be represented conceptually as The fractional order determines the structure of the historical weighting. This makes fractional calculus suitable for modelling systems with memory and hereditary behaviour.

IV. Research Motivation

Existing research has established several definitions and properties of nabla fractional sums and differences. These include transform methods, product rules, summation-by-parts formulas, and applications to fractional difference equations.

However, from a computational and conceptual perspective, it is useful to explicitly represent the historical information as a vector and the fractional operation as a projection.

This paper proposes such a representation. The main idea is

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V. Proposed Projection-Based Framework

Let the historical state vector at time be Define the fractional projection weights by the proposed fractional projection is where represents the contribution of the historical state. Using the nabla fractional-sum kernel, we obtain the projection interpretation provides a direct connection between historical data and fractional memory.

VI. Fractional Memory Projection

We introduce the term Fractional Memory Projection (FMP) for the proposed representation.

Define Then where:

is the present discrete instant;

is a historical instant;

measures the age of the historical information;

determines the fractional memory structure;

represents the memory weight.

Therefore, the fractional state is obtained through the projection of historical information onto a fractional memory direction.

VII. Proposed Fractional Historical Projection Principle

The principal proposed principle can be stated as follows:

For a discrete sequence, its fractional state at time can be represented as the projection of its historical state vector onto a fractional memory vector determined by the fractional order. Mathematically, where this provides the relationship

VIII. Normalized Projection

For numerical applications, normalized weights can be defined as Consequently, the normalized projection is and can be interpreted as a fractional-memory weighted representation of historical observations.

IX. Projection Matrix Representation

For a discrete sequence the proposed projection can be represented by a lower-triangular matrix: Therefore, provides a computational representation of the fractional projection. Each row represents the contribution of historical observations to the corresponding discrete state.

X. Illustrative Example Consider

At, the historical vector is for the fractional projection incorporates all available historical observations:

Unlike an ordinary first-order difference, the fractional representation therefore retains historical information.

The example demonstrates the fundamental concept of the proposed projection framework.

XI. Advantages

The proposed projection framework has several potential advantages.

11.1 Explicit memory representation

Historical observations are explicitly represented in the model.

11.2 Computational implementation

The matrix formulation permits implementation using standard numerical linear algebra.

11.3 Data-driven modelling

The projection can be applied directly to discrete observations.

11.4 Adjustable memory

The fractional order controls the memory structure.

11.5 Extension to vector systems

For the framework can be extended to multivariable systems.

XII. Possible Applications

The proposed framework may be investigated in:

Time-series analysis – modelling historical dependence in discrete observations.

Digital signal processing – fractional filtering and memory-dependent signal transformation.

Control systems – discrete controllers incorporating historical states.

Biological modelling – systems with hereditary effects.

Economic modelling – processes with long-range dependence.

Numerical analysis – computational solutions of fractional difference equations.

Discrete dynamical systems – analysis of memory-dependent discrete processes.

XIII. Discussion

The proposed approach should be distinguished from the established definitions of nabla fractional operators. The mathematical kernel used in the projection is closely related to known nabla fractional sums.

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Therefore, the novelty claimed here is primarily the projection-based formulation and interpretation, rather than the invention of a fundamentally new fractional derivative.

This distinction is important for responsible publication. To establish stronger mathematical novelty, future research should investigate whether the projection operator possesses properties that are not immediately available from existing formulations. Important questions include:

What are the boundedness conditions?

Under what conditions is the projection stable?

What happens as?

Can an optimal be estimated from data?

Can the projection be extended to variable-order systems?

Can convergence and approximation theorems be established?

XIV. Variable-Order Extension

A natural extension is to allow the fractional order to vary with time:

The proposed operator becomes This provides a model in which the memory structure changes dynamically. Such an approach may be useful for systems whose historical dependence is not constant over time.

XV. Future Scope

Future research can develop the proposed framework in four major directions.

First, rigorous mathematical properties such as linearity, boundedness, stability, and convergence can be established. Second, numerical algorithms can be developed for efficient computation of the projection when the historical sequence becomes large. Third, the fractional order can be estimated from experimental or observed data. Fourth, numerical experiments can compare the proposed projection with conventional nabla fractional sums, Caputo-type differences, and other discrete fractional formulations. Such investigations would determine whether the projection framework provides measurable advantages over existing methods.

XVI. Conclusion

Nabla fractional differential calculus provides an important mathematical framework for discrete systems with nonlocal and memory-dependent behaviour.

This paper has presented a projection-based framework in which the historical states of a discrete system are combined with fractional memory weights to obtain a projected fractional state.

The proposed representation is and can also be expressed in matrix form as the framework offers a possible bridge between discrete fractional calculus, numerical computation, and data-driven modelling.

However, further mathematical proofs and numerical experiments are required before the projection framework can be presented as a fully established new theory. This provides a clear direction for future research and possible development into a stronger original research article.

Footnotes

1. The nabla operator is the backward difference operator commonly used in discrete calculus and time-scale calculus.
2. The Gamma function extends the factorial function to non-integer arguments and is fundamental to fractional calculus.
3. The rising factorial is naturally associated with kernels appearing in discrete fractional sums.
4. Different definitions of discrete fractional operators may use different starting points, domains, and conventions.
5. The projection-based framework proposed here requires additional theoretical and numerical validation before its novelty and advantages can be conclusively established.

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