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Analytical Study on Semi - Bounded Linear Operators in Sesquilinear forms of Hilbert Spaces

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Abstract-- **This Research Paper presents the Study of Semi - Bounded Linear Operators in Sesquilinear forms of Hilbert Spaces. Here, we discuss Spectral gaps, the study of Semi-Bounded perturbations in quantum Mechanics on their Spectral properties, domain extensions, applications in quantum system and partial differential equations. In this paper, we prove the evolving role of semi-bounded linear operators in both theoretical and applied mathematical physics in terms of complex systems in various domains.**

Keywords-- **Semi-Bounded Linear operators, Sesquilinear forms, Hilbert Spaces, Spectral theory, Quantum Mechanics, Domain extensions.**

I. INTRODUCTIONS

Kato, T.(2) is the pioneer worker of the present area, In fact, the present work is the extension of work done by Davies, E.B.(1), Kothe, G. et al (3,4), Reed, M. et al (5), Ghosh, H.C. et al(6), Srivastava et al (7) , Srivastava et al. (8) and Yadav S.K. et al. (09) In this paper, we have studied Analytically Semi- Bounded Linear operators in Sesquilinear forms of Hilbert Spaces.

Here, we use the following preliminaries, Notations and fundamental ideas:-

II. PRELIMINARIES

Sesquilinear Forms Definition

A sesquilinear form a on a Hilbert space H is a mapping:

a: D (a) × D (a) → C, $a(x,y) = \langle x,y \rangle H, for\ x,y \in D\ (a)$,

Where D (a) ⊆ H is the domain of the form, and $\langle x, \rangle H$ denotes the inner product in the Hilbert space H.

Semi - Bounded Operators Definition

An operator A is semi - bounded if there exists a constant $c \in R$ such that:

$$\langle Ax, \rangle \geq c \parallel x \parallel 2\ \forall x \in D(A),$$

Where

$\langle Ax,x \rangle$ *represents the action of* A *on* x *and* $D\ (A)$*is the domain of the operator.*

Connections between Sesquilinear Forms and Operators

For any densely defined, closed, semi - bounded sesquilinear form a, there exists a unique self-adjoint, semi-bounded operator A such that:

$$A\ (x, y) = \langle Ax, y \rangle, \forall x, y \in D\ (A)$$

Spectral Theorem for Self-Adjoint Operators

For a self-adjoint operator A, the spectral theorem provides a decomposition of the operator in terms of its eigenvalues and eigenvectors. The operator A written as:

$$A = \int \sigma(A)\ , \lambda dE(\lambda)$$

Eigen value Equation for Schrödinger Operator

The Schrödinger operator H in quantum mechanics is typically bounded from below. The Eigen value equation for H is:

$$H\psi_n = E_n\psi_n,$$

where H is the Hamiltonian operator, ψ_n are the Eigen functions (quantum states), and E_n are the corresponding Eigen values (energy levels).

Domain Extension of Semi-Bounded Operators

When extending a semi-bounded operator A in a Sobolev space, the extended operator $\tilde{A}$ satisfies:

$$\breve{A} \supset A \text{ with } \langle \breve{A}x, \rangle \geq c \parallel x \parallel 2, \forall x \in D(\tilde{A})$$

Perturbation Theory for Semi-Bounded Operators

In quantum mechanics, perturbation theory is used to study the effect of small disturbances on a bounded operator H_0. The perturbed Hamiltonian H written as:

$$H = H_0 + \lambda V,$$

Where H_0 is the unperturbed Hamiltonian, λ is a small perturbation parameter, and V is the perturbing operator.

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Extension Theory of Semi-Bounded Operators

Generalized Domains: One emerging trend is the development of extension theory for semi-bounded operators, where a class of extensions of these operators is studied.

Self-Adjoint Extensions: In quantum mechanics and Partial Differential Equations (PDEs), operators are often semi-bounded and need to be self-adjoint for the well-defined evolution of systems.

Spectral Theory of Semi-Bounded Operators

Spectral Gaps: A significant area of current research in semi-bounded operators involves the study of spectral gaps—regions where the spectrum does not contain any eigenvalues.

Asymptotics of Spectra:

This involves analyzing the rate of growth of eigenvalues and understanding the operator's semi-bounded nature influences the distribution of eigenvalues.

Applications in PDEs and Mathematical Physics

Non-Linear Semi-Bounded Operators: The study of non-linear semi-bounded operators, particularly in the context of elliptic partial differential equations (PDEs), is an emerging trend. These operators are used in the study of reaction-diffusion equations, nonlinear Schrödinger equations, and other nonlinear wave equations.

Boundary-Value Problems: Semi-Bounded operators are also being studied in the context of boundary-value problems for PDEs.

Operator-Valued Semi-Bounded Operators

Matrix-Valued Functions: In the study of Semi-Bounded operators, operator-valued functions, such as matrix-valued measures or matrix functions, are increasingly being considered. The extension of classical results from scalar to matrix-valued operators is a key area of current research, particularly in the context of multi-dimensional PDEs and systems with multiple degrees of freedom.

Applications in Quantum Systems and Signal Processing: Semi-Bounded matrix operators are being applied to multi-dimensional systems in quantum mechanics and signal processing, especially in the analysis of multi-mode systems where interactions between modes are semi-bounded.

III. Domain Extensions And Form Closures

In the context of semi-bounded operators, especially in quantum mechanics and partial differential equations (PDEs), the concepts of domain extensions and form closures are essential for ensuring that operators are well-defined and exhibit desirable properties like self-adjointness, spectral properties, and stability under various perturbations. The recent trend of generalizing these concepts focuses on accommodating irregular boundary conditions and singular potentials, which are common in complex physical and mathematical systems.

Domain Extensions of Semi-Bounded Operators

The domain of an operator refers to the set of functions or vectors on which the operator acts. In the case of semi-bounded operators, this domain is crucial because it helps define the operator's behavior, especially its spectral properties. Domain extensions refer to the process of extending the domain of an operator to include a broader class of functions or distributions, often to deal with irregularities like singular potentials or boundary conditions that the original domain cannot accommodate.

Irregular Boundary Conditions:

In many applications, operators are subject to boundary conditions that may not be smooth or regular. These boundary conditions can lead to challenges in defining operators, especially when dealing with unbounded domains. The classical domain of an operator may not be sufficient for such problems, requiring an extension of the operator's domain to functions that behave in a more irregular manner at the boundary.

Singular Potentials: Potentials in quantum mechanics, such as the Coulomb potential in the hydrogen atom, can introduce singularities. When dealing with such potentials, the standard approach might not be well-defined, and one might need to extend the domain to accommodate functions with singular behavior near these potential singularities.

Form Closures:

In operator theory, the form closure refers to the process of extending a form domain so that the operator becomes self-adjoint or well-defined in a larger space. This is necessary when the initial domain of the form is not sufficient to describe all possible behaviors of the operator, particularly under boundary conditions or singularities.

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A closed form allows the operator to be extended to a self-adjoint operator, which is important for ensuring that its spectrum is real and that it has desirable properties such as a well-defined evolution in quantum mechanics.

Techniques in Form Closures:

Generalized Forms: Recent research focuses on generalizing forms to accommodate more irregular situations, especially when dealing with singular potentials.

Non-Smooth Functions: Another aspect of recent research is the consideration of nonsmooth functions for the form domain. By extending the form closure process to functions that are not smooth (e.g., discontinuous or having singularities), researchers handle more complex systems where traditional smoothness assumptions do not hold.

Recent Techniques and Challenges

Recent trends in the generalization of domain extensions and form closures focus on making these concepts more flexible and applicable to real-world problems that involve irregular boundary conditions or singular potentials.

a) Weighted Sobolev Spaces are often employed to extend the domain of operators that encounter singular behaviour. In these fields, weight functions enable the management of functions' behaviors close to either boundaries or singular potentials. More complex operators deal with that might otherwise be ill-defined
b) The techniques of the non-standard variation procedure alter the standard variational formulation for the characterization of problems exhibiting non-standard behaviour particularly singular potentials
c) Broadening form closures to encompass spaces of non-smooth functions enables the operator theory to address singularities that arise in physical and mathematical contexts.

Applications and Impact

In quantum mechanics, semi-bounded operators like the Hamiltonian have applications because of singular potentials like the Coulomb potential or irregular boundaries like those at infinity. In order to ensure that operators are mathematically consistent, we employ domain extensions and generalizations of form domains that yield the desired operator with well-defined spectral properties. Extensions of domains and closures of forms in the context of PDEs allows for other possible operators that don't necessarily satisfy local conditions. Thus enabling us to solve for PDEs with irregular or singular boundary data. This method is especially useful in mathematical physics, including general relativity and electromagnetism, where potentials may be singular in certain regions.

For several years, domain extensions (in sense of continuous operators) and closures (in weak sense) of semi-bounded operators are under the observation of researchers. The merit of this zone is its ability to deal with irregular boundary conditions and singularities. Weighted Sobolev spaces, nonstandard variational methods, and the extension of the form closure process to non-smooth functions are techniques developed to generalize operator theory.

Mathemtical Developments.

For some time, there has been work on the notion of essential self-adjointness, including its precise definition. It is now possible to develop a more general criterion that allows one to declare essential self-adjointness for an operator defined on a non-dense domain. This is to say the usage of boundary triples, which are mathematical gadgets that relate unbounded operators to their adjoints.

A symmetric operator is self-adjoint if it has deficiency indices (0, 0)

Given a densely defined symmetric operator AA on a Hilbert space, its adjoint A∗A^* is generally a larger operator. Here, AA is essentially self-adjoint (i.e., whether its closure is self-adjoint), we consider the deficiency spaces:

$$k_{\pm} = \ker(A^* \mp iI)$$

and define the deficiency indices as:

$n_+ = \dim K_+ n_- = dim K_-$

These count the number of linearly independent solutions to the equations:

$$(A* - iI)\psi = 0 \text{ } and \text{ } (A* + iI)\psi = 0$$

Key result (von Neumann's theorem):

- If n+=n−=0n_+ = n_- = 0, then **AA is essentially self-adjoint.**
- If n+=n−>0n_+ = n_- > 0, then **AA has infinitely many self-adjoint extensions.**
- If n+≠n−n_+ \ne n_-, then **AA has no self-adjoint extensions.**

In quantum mechanics, the framework consists of quantum states, observables and quantities, and measurement. Should the essential self-adjointness be guaranteed, the predictions of the observable are unique.

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In Quantum Mechanics it deals with the Hamiltonian operator whose domain is not dense. This is especially the case for systems with singular potentials or boundary conditions. We modify the self-adjointness condition in these cases so that the spectrum is real and discrete.

The system of boundary triples involves extending the operator to a larger space with controlled boundary conditions. This helps to identify self-adjoint extensions (or boundary problems) of the operator.

Methods and Uses.

The spectrum of Schrödinger operators with singular or irregular potentials, like the hydrogen atom whose potential has a singularity at the origin, has been extensively studied via Friedrichs extensions. The extensions offer a systematic means to manage these singularities while ensuring that the spectrum is real and discrete.

In spectral theory, we wish to decompose an operator into its spectral parts. We use the Friedrichs extensions in order to make the operator self-adjoint and give it a well-behaved spectrum.

Use of quantum fields and many-body systems – applications to Hydrogen atom, hydrogen like atom and hydrogen atom with magnetic field. In quantum field theory and many-body physics, Friedrichs extensions are used to study Hamiltonians. It enables a more accurate comprehension of the energy levels and stability of complex systems.

All in all, recent advancements in spectral analysis of semi-bounded operators have greatly improved our ability to manage intricate systems in quantum mechanics and mathematical physics. By refining the criteria for essential self-adjointness on non-dense domains, a broader class of operators to become spectral even in the presence of irregular boundary conditions or singular potentials. The use of Friedrichs extensions facilitates the process of proving the self-adjointness of semi-bounded operators. One particular application of this result is the case of Schrödinger operators with singular potentials, which we will now discuss in detail. The developments above will have important consequences for quantum mechanics, PDEs and other fields in which semi-bounded operators play an important role.

IV. Mathematical Physics Applications

The modeling of quantum systems with semi-bounded operators is crucial, especially when unbounded energy operators are involved. Recent studies have contributed significantly to understanding and applying these operators in various domains. Notable advancements include:

• *Eigenvalue Calculation Strategies:* New methods for computing the eigenvalues of bounded operators are being developed, with a focus on ensuring their accuracy and relevance in complex quantum systems.

• *Sesquilinear Forms in Non-Linear Quantum Systems:* Semi-Bounded operators are increasingly used in the context of non-linear quantum systems. The use of sesquilinear forms allows for the proper definition and manipulation of operators in these systems, enabling the study of more complex and realistic quantum models.

V. Future Direction & Conclusion:-

On the basis of the above preliminaries, fundamental ideas and mathematical analysis, In this paper we arrive at the New Zones as

(i) Non-linear Hilbertian space with sesquilinear forms.

(ii) Development of techniques to numerically approximate the spectra of semi-bounded operators.

(iii) Applications of machine learning through semi-bounded operators in the context of kernel methods.

Thus, this paper indicates the New Characterization of the study of semi-bounded operators arising from sesquilinear forms defined in Hilbert spaces. By using up-to-date technologies, researchers solve complex problems in functional analysis and mathematical physics.

Hence the result.

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REFERENCES

[1] Davies, E. B. (1905) : Spectro Theory and Differential operators, Cambridge University Press.

[2] Kato, T. (1980): Perturbation Theory for Linear operators, Springer.

[3] Kothe, G. (1969): Topological Vector Spaces, Springer – Verlag, I, New York.

[4] Kothe, G. (1979):Topological Vector Spaces, Springer – Verlag, II, New York.

[5] Reed, M; and Simon, B. (1972): Methods of Modern Mathematical Physics, Academic, Press.

[6] Ghosh, H.C.; Yadav, S.K., Kumar, Abhishek & Srivastava, U.K. (2022) : U.S.A., Vol. 08 (07), pp. 45 – 50.

[7] Srivastava, U.K.; Kumar, S., Kumar, N and Singh, T.N. (2012): Proc. of Math. Soc. B.H.U. Vol. 28, pp. 25 – 28.

[8] Srivastava, U.K.; and Singh, T.N. (2014):International Journal of Emerging Technology and Advanced Engineering, Vol. 04 (03), pp. 658 – 659.

[9] Yadav, S.K.; Thakur, K.B. and Srivastava, U.K. (2024): International Journal of Recent Development in Engineering and Technology, Vol. 13(06), p.p. 49 – 51.