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A Comprehensive Study of Fourier Series for the Generalized I-Function

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***Abstract*— This paper presents a comprehensive study on the Fourier series expansions of the generalized I-function of several complex variables. By evaluating two integrals involving the I-function, we derive the corresponding Fourier sine and cosine series representations. These results generalize and extend existing work on the H-function and I-function, incorporating earlier findings by N. Bhati and R.K. Gupta as particular cases. The analytical approach leverages Mellin–Barnes contour integrals and orthogonality properties of trigonometric functions to establish convergence and derive explicit series expressions. The outcomes of this study are significant in the broader application of special functions in mathematical analysis and theoretical physics.**

***Keywords*— I-function, Fourier series, H-function, Mellin–Barnes integral, generalized special functions, multiple complex variables, sine series, cosine series**.

I. INTRODUCTION

Fourier series plays a crucial role in environmental science and sustainability by analyzing and modeling periodic phenomena in nature. It is widely used in climate studies to identify seasonal patterns in temperature, precipitation, and greenhouse gas emissions, aiding in climate change predictions. In water and air pollution monitoring, Fourier series helps detect cyclical trends in pollutant levels, enabling better regulatory strategies. It also supports oceanography by analyzing tidal movements and wave patterns, which is essential for coastal management and renewable energy generation from tides. Additionally, in noise pollution control, Fourier analysis helps in filtering and reducing environmental noise, improving urban sustainability. By applying Fourier series in these areas, researchers can develop data-driven solutions for environmental conservation and sustainable development.

Rathie's work in 1997 expanded the scope of Fox's H-function, a special function that has recently attracted considerable attention due to its applications in wireless communication systems [5–7]. Later, Shantha Kumari, Nambisan, and Rathie advanced this line of research by formulating the I-function of two variables, which naturally extends the two-variable H-function earlier introduced by Mittal and Gupta, while also establishing several of its fundamental properties.

In this study, we take the next step by generalizing the I-function to encompass *r* variables. This broader formulation may be viewed as a natural continuation of the H-function of *r* variables originally proposed by Srivastava and Panda.

II. I-FUNCTION OF SEVERAL VARIABLES

The generalized Fox H-function, namely, I-function of "r" variables, is defined and represented by Prathima, Nambisan and Shantha Kumari [8] in the following manner:

$$I[z_1,z_2,\ldots,z_r] = I^{0,N:m_1,n_1;\ldots;m_r,n_r}_{P,Q:p_1,q_1;\ldots;p_r,q_r}\left[z_1\ldots z_r\left|\begin{matrix}\left(a_j;\alpha_j^{(1)}\ldots\alpha_j^{(r)};A_j\right)_{1,p}:\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{1,p_1}\\ \left(b_j;\beta_j^{(1)}\ldots\beta_j^{(r)};B_j\right)_{1,q}:\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{1,q_1}\end{matrix}\right.\right.$$

$$\left.\begin{matrix}\ldots\left(c_j^{(r)},\gamma_j^{(r};C_j^{(r)}\right)_{1,p_r}\\ \ldots\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r}\end{matrix}\right]$$

$$= \frac{1}{(2\pi i)^r}\int_{\lambda_1}\cdots\int_{\lambda_r}\Phi_1(s_1),\ldots,\Phi_r(s_r)\Psi(s_1,s_2,\ldots,s_r)z_1^{s_1}\ldots z_r^{s_r}ds_1\ldots ds_r, \quad \ldots(1)$$

where $i=1,\ldots,r \; and \; \Phi_i(s_i),\Psi(s_1,\ldots,s_r)$ are given by

$$\Psi(s_1,\ldots,s_r)=\frac{\prod_{j=1}^{N}\Gamma^{A_j}(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}s_i)}{\prod_{j=N+1}^{P}\Gamma^{A_j}(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}s_i).\prod_{j=1}^{Q}\Gamma^{B_j}(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}s_i)},$$

…(2)

$$\Phi_i(s_i)=\frac{\prod_{j=1}^{m_i}\Gamma^{D_j}(d_j-\sum_{i=1}^{r}\delta_j^{(i)}s_i).\prod_{j=1}^{n_i}\Gamma^{C_j}(1-c_j+\sum_{i=1}^{r}\gamma_j^{(i)}s_i)}{\prod_{j=n_i+1}^{p_i}\Gamma^{C_j}(c_j-\sum_{i=1}^{r}\gamma_j^{(i)}s_i).\prod_{j=m_i+1}^{q_i}\Gamma^{D_j}(1-d_j+\sum_{i=1}^{r}\delta_j^{(i)}s_i)}, \quad \ldots(3)$$

where i = 1, ..., r.

Let the parameters m_j , n_j , p_j , q_j (j = 1, ..., r), along with N, P and Q be non-negative integers subject to the conditions $0 \leq N \leq P$, $Q \geq 0$, $0 \leq n_j \leq p_j$, and $0 \leq m_j \leq q_j$, with the restriction that they are not all zero at the same time..

For normalization purposes, the quantities indexed as $\alpha_j^{(i)}$ (j = 1, ..., P, i = 1, ..., r), $\beta_j^{(i)}$ (j = 1, ..., Q, i = 1, ..., r), $\gamma_j^{(i)}$ (j = 1, ..., p_i, i =1, ..., r) and $\delta_j^{(i)}$ (j = 1, ..., Q, i = 1, ..., r) are assumed to be positive.

The sets a_j (j = 1, ..., P), b_j (j = 1, ...,Q), $c^{(i)}_j$ (j = 1, ..., p_i, i = 1, ..., r) and $d^{(i)}_j$ (j = 1, ..., q_i, i = 1, ..., r) denote complex parameters. Similarly, the coefficients A_j (j = 1, ..., P),B_j (j = 1, ...,Q), $C^{(i)}_j$ (j = 1, ..., p_i, i = 1, ..., r) and $D^{(i)}_j$ (j = 1, ..., q_i, i= 1, ..., r) appearing in the gamma functions of equations (2) and (3) are not restricted to integer values.

Furthermore, each contour γ_i in the complex s_i-plane follows the Mellin–Barnes type path, extending from $c - i\infty$ to $c + i\infty$ with an indentation.

In accordance with Braaksma , the I-function in several variables is analytic under these conditions.

$$\mu_i = \sum_{j=1}^{P} A_j\alpha_j^{(i)} - \sum_{j=1}^{Q} B_j\beta_j^{(i)} + \sum_{j=1}^{p_i} C_j\gamma_j^{(i)} - \sum_{j=1}^{q_i} D_j\delta_j^{(i)} \leq 0, i = 1,\ldots,r.$$

…(4)

Integral (1) converges absolutely if,

$$\left|\arg(z_k)\right| < \frac{1}{2}\Delta_k\pi, i = 1,\ldots,r. \qquad \ldots(5)$$

Where,

$$\Delta_k = \left[-\sum_{j=N+1}^{P} A_j\alpha_j^{(k)} - \sum_{j=1}^{Q} B_j\beta_j^{(k)} + \sum_{j=1}^{n_k} C_j^{(k)}\gamma_j^{(k)} + \sum_{j=1}^{m_k} D_j^{(k)}\delta_j^{(k)} - \sum_{j=n_k+1}^{p_k} c_j^{(k)}\gamma_j^{(k)} - \sum_{j=m_k+1}^{q_k} d_j^{(k)}\delta_j^{(k)} \right] > 0, k = 1,\ldots,r.$$

…(6)

III. PRELIMINARIES

From the table of integrals Gradshteyin and Ryzhik[10],we are required the following formulae in our proof:

$$\int_0^{\pi} (\sin\theta)^{2\mu} \sin(2n+1)\theta d\theta = \frac{(-1)^n\sqrt{\pi}\,\Gamma\left(1/2+\mu\right)\Gamma(1+\mu)}{\Gamma\left(1/2+\mu-n\right)\Gamma\left(3/2+\mu+n\right)}, \mathrm{Re}(\mu) > -1/2,$$

…(7)

$$\int_0^{\pi} (\sin\theta)^{2\mu} \cos(2n\theta) d\theta = \frac{(-1)^n\sqrt{\pi}\,\Gamma\left(1/2+\mu\right)\Gamma(1+\mu)}{\Gamma(1+\mu-n)\Gamma(1+\mu+n)}, \mathrm{Re}(\mu) > -1/2,$$

...(8)

IV. MAIN RESULTS

This section derives specific integrals by applying the results established in equations (7) and (8).

A. First Main Integral:

$$\int_0^{\pi} (\sin\theta)^{2\mu} \sin(2n+1)\theta\, I^{0,N:m_1,n_1;\ldots;m_r,n_r}_{P,Q:p_1,q_1;\ldots;p_r,q_r}$$

$$\left[z_1(\sin\theta)^{2\lambda_1} \ldots z_r(\sin\theta)^{2\lambda_r} \left| \begin{matrix} (a_j;\alpha_j^{(1)}\ldots\alpha_j^{(r)};A_j)_{1,P}: \\ (b_j;\beta_j^{(1)}\ldots\beta_j^{(r)};B_j)_{1,Q}: \end{matrix} \right. \begin{matrix} (c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,p_1}\ldots(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,p_r} \\ (d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,q_1}\ldots(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,q_r} \end{matrix} \right] d\theta =$$

$$(-1)^n\sqrt{\pi}\, I^{0,N+2:m_1,n_1;\ldots;m_r,n_r}_{P+2,Q+2:p_1,q_1;\ldots;p_r,q_r}$$

$$\left[z_1\ldots z_r \left| \begin{matrix} (-\mu+1/2;\lambda_j^i,\ldots,\lambda_j^r);(-\mu;\lambda_j^i,\ldots,\lambda_j^r); \\ (b_j;\beta_j^{(1)}\ldots\beta_j^{(r)};B_j)_{1,Q};(-\mu\pm 1/2\pm n;\lambda_j^i,\ldots,\lambda_j^r); \end{matrix} \right. \right.$$

$$\left. \begin{matrix} (a_j;\alpha_j^{(1)}\ldots\alpha_j^{(r)};A_j)_{1,P+2};(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,p_1}\ldots(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,p_r} \\ (d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,q_1}\ldots(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,q_r} \end{matrix} \right]$$

…(9)

The conditions for convergence are same as above eq. (5) and (6).

B. Second Main Integral:

$$\int_0^{\pi} (\sin\theta)^{2\mu} \cos(2n\theta)\, I^{0,N:m_1,n_1;\ldots;m_r,n_r}_{P,Q:p_1,q_1;\ldots;p_r,q_r}$$

$$\left[z_1(\sin\theta)^{2\lambda_1} \ldots z_r(\sin\theta)^{2\lambda_r} \left| \begin{matrix} (a_j; \alpha_j^{(1)} \ldots \alpha_j^{(r)}; A_j)_{1,P} : \\ (b_j; \beta_j^{(1)} \ldots \beta_j^{(r)}; B_j)_{1,Q} : \end{matrix} \right. \right.$$

$$\left. \begin{matrix} (c_j^{(1)}, \gamma_j^{(1)}; C_j^{(1)})_{1,p_1} \ldots (c_j^{(r)}, \gamma_j^{(r}; C_j^{(r)})_{1,p_r} \\ (d_j^{(1)}, \delta_j^{(1)}; D_j^{(1)})_{1,q_1} \ldots (d_j^{(r)}, \delta_j^{(r)}; D_j^{(r)})_{1,q_r} \end{matrix} \right] d\theta =$$

$$(-1)^n \sqrt{\pi} I^{0,N+2:m_1,n_1;\ldots;m_r,n_r}_{P+2,Q+2:p_1,q_1;\ldots;p_r,q_r}$$

$$\left[z_1 \ldots z_r \left| \begin{matrix} (-\mu + 1/2; \lambda_j^{i}, \ldots, \lambda_j^{r}); (-\mu; \lambda_j^{i}, \ldots, \lambda_j^{r}); \\ (b_j; \beta_j^{(1)} \ldots \beta_j^{(r)}; B_j)_{1,Q+2}; (-\mu \pm n; \lambda_j^{i}, \ldots, \lambda_j^{r}); \end{matrix} \right. \right.$$

$$\left. \begin{matrix} (a_j; \alpha_j^{(1)} \ldots \alpha_j^{(r)}; A_j)_{1,P+2}; (c_j^{(1)}, \gamma_j^{(1)}; C_j^{(1)})_{1,p_1} \ldots (c_j^{(r)}, \gamma_j^{(r}; C_j^{(r)})_{1,p_r} \\ (d_j^{(1)}, \delta_j^{(1)}; D_j^{(1)})_{1,q_1} \ldots (d_j^{(r)}, \delta_j^{(r)}; D_j^{(r)})_{1,q_r} \end{matrix} \right]$$

…(10)

The conditions for convergence are same as above eq. (5) and (6).

V. PROOF OF MAIN RESULTS

To derive the First Main Integral (9), we begin by utilizing the known result (7), which is already presented in integral form. Next, we represent the I-function of r-variables using the Mellin-Barnes type contour integral as outlined on the right-hand side of result (1). By invoking the property of absolute convergence, we are able to interchange the order of integration. With further simplification, this leads us directly to the desired result

VI. PARTICULAR CASES

Assigning the value one to each exponent A_j (j = 1, ..., P), B_j (j = 1, ...,Q),$C_j^{(i)}$ (j = 1, ..., p_i, i = 1, ..., r) and $D_j^{(i)}$ (j = 1, ..., qi, i = 1, ..., r) in equations (9) and (10) transforms the I-function into the H-function of multiple variables as defined by Srivastava and Panda . The corresponding results are consistent with those obtained by N. Bhati and R. K. Gupta.

A. Fourier Sine Series of I-Function:

The Fourier sine series represents a function as an infinite summation of sine terms and is especially effective for expressing odd functions defined over a finite interval. For the I-function, this series can be used to break down the function into sine components, with coefficients calculated through integrals involving the I-function itself. This decomposition provides a structured way to analyze and approximate the I-function through simpler sinusoidal elements, which is important for solving boundary value problems and has wide-ranging applications in mathematical physics and engineering. Leveraging the orthogonality of sine functions, the Fourier sine series form of the I-function aids in exploring its characteristics and behavior across multiple variables.

$$(\sin\theta)^{2\mu} I^{0,N:m_1,n_1;\ldots;m_r,n_r}_{P,Q:p_1,q_1;\ldots;p_r,q_r}$$

$$\left[z_1(\sin\theta)^{2\lambda_1} \ldots z_r(\sin\theta)^{2\lambda_r} \left| \begin{matrix} (a_j; \alpha_j^{(1)} \ldots \alpha_j^{(r)}; A_j)_{1,P}; \\ (b_j; \beta_j^{(1)} \ldots \beta_j^{(r)}; B_j)_{1,Q}; \end{matrix} \right. \right.$$

$$\left. \begin{matrix} (c_j^{(1)}, \gamma_j^{(1)}; C_j^{(1)})_{1,p_1} \ldots (c_j^{(r)}, \gamma_j^{(r}; C_j^{(r)})_{1,p_r} \\ (d_j^{(1)}, \delta_j^{(1)}; D_j^{(1)})_{1,q_1} \ldots (d_j^{(r)}, \delta_j^{(r)}; D_j^{(r)})_{1,q_r} \end{matrix} \right]$$

$$= \sum_{v=0}^{\infty} \frac{2(-1)^v}{\sqrt{\pi}} \sin(2v+1)\theta \, . \, I^{0,N+2:m_1,n_1;\ldots;m_r,n_r}_{P+2,Q+2:p_1,q_1;\ldots;p_r,q_r}$$

$$\left[z_1 \ldots z_r \left| \begin{matrix} \left(-\mu + \frac{1}{2; \lambda_j^{i}}, \ldots, \lambda_j^{r}\right); (-\mu; \lambda_j^{i}, \ldots, \lambda_j^{r}); \\ (b_j; \beta_j^{(1)} \ldots \beta_j^{(r)}; B_j)_{1,Q+2}; (-\mu \pm n; \lambda_j^{i}, \ldots, \lambda_j^{r}); \end{matrix} \right. \right.$$

$$\left. \begin{matrix} (a_j; \alpha_j^{(1)} \ldots \alpha_j^{(r)}; A_j)_{1,P+2}; (c_j^{(1)}, \gamma_j^{(1)}; C_j^{(1)})_{1,p_1} \ldots (c_j^{(r)}, \gamma_j^{(r}; C_j^{(r)})_{1,p_r} \\ (d_j^{(1)}, \delta_j^{(1)}; D_j^{(1)})_{1,q_1} \ldots (d_j^{(r)}, \delta_j^{(r)}; D_j^{(r)})_{1,q_r} \end{matrix} \right] , R(2\mu) \geq 0, 0 \leq \theta \leq \pi$$

B. Fourier Cosine Series of I-Function:

The Fourier cosine series expresses a function as an infinite sum of cosine terms and is particularly suitable for representing even functions on a finite interval. When applied to the I-function, the Fourier cosine series allows the function to be expanded into a series of cosine functions, where the coefficients are obtained by integrating the product of the I-function and cosine terms. This approach facilitates the analysis and approximation of the I-function using basic oscillatory components, which is beneficial in solving various boundary value and partial differential equations. By utilizing the orthogonality of cosine functions, the Fourier cosine series representation helps in examining the properties and behavior of the I-function in multiple dimensions, offering valuable tools for applications in applied mathematics, physics, and engineering.

$$(\sin\theta)^{2\mu} I^{0,N:m_1,n_1;\ldots;m_r,n_r}_{P,Q:p_1,q_1;\ldots;p_r,q_r}$$

$$\left[z_1(\sin\theta)^{2\lambda_1} \ldots z_r(\sin\theta)^{2\lambda_r} \,\middle|\, \begin{array}{l} \left(a_j;\alpha_j^{(1)}\ldots\alpha_j^{(r)};A_j\right)_{1,P}; \\ \left(b_j;\beta_j^{(1)}\ldots\beta_j^{(r)};B_j\right)_{1,Q}; \end{array} \right.$$
$$\left. \begin{array}{l} \left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{1,p_1}\ldots\left(c_j^{(r)},\gamma_j^{(r};C_j^{(r)}\right)_{1,p_r} \\ \left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{1,q_1}\ldots\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r} \end{array} \right]$$

$$= \sum_{v=0}^{\infty} \frac{(-1)^v}{\sqrt{\pi}} . I^{0,N+2:m_1,n_1;\ldots;m_r,n_r}_{P+2,Q+2:p_1,q_1;\ldots;p_r,q_r}$$

$$\left[z_1\ldots z_r \,\middle|\, \begin{array}{l} \left(-\mu+1/2;\lambda_j^i,\ldots,\lambda_j^r\right);\left(-\mu;\lambda_j^i,\ldots,\lambda_j^r\right); \\ \left(b_j;\beta_j^{(1)}\ldots\beta_j^{(r)};B_j\right)_{1,Q+2};\left(-\mu\pm n;\lambda_j^i,\ldots,\lambda_j^r\right); \end{array} \right.$$
$$\left. \begin{array}{l} \left(a_j;\alpha_j^{(1)}\ldots\alpha_j^{(r)};A_j\right)_{1,P+2};\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{1,p_1}\ldots\left(c_j^{(r)},\gamma_j^{(r};C_j^{(r)}\right)_{1,p_r} \\ \left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{1,q_1}\ldots\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r} \end{array} \right]$$
$$+\sum_{v=0}^{\infty} \frac{2(-1)^v}{\sqrt{\pi}} \cos v\theta . I^{0,N+2:m_1,n_1;\ldots;m_r,n_r}_{P+2,Q+2:p_1,q_1;\ldots;p_r,q_r}$$

$$\left[z_1\ldots z_r \,\middle|\, \begin{array}{l} \left(-\mu+1/2;\lambda_j^i,\ldots,\lambda_j^r\right);\left(-\mu;\lambda_j^i,\ldots,\lambda_j^r\right); \\ \left(b_j;\beta_j^{(1)}\ldots\beta_j^{(r)};B_j\right)_{1,Q+2};\left(-\mu\pm n;\lambda_j^i,\ldots,\lambda_j^r\right); \end{array} \right.$$
$$\left. \begin{array}{l} \left(a_j;\alpha_j^{(1)}\ldots\alpha_j^{(r)};A_j\right)_{1,P+2};\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{1,p_1}\ldots\left(c_j^{(r)},\gamma_j^{(r};C_j^{(r)}\right)_{1,p_r} \\ \left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{1,q_1}\ldots\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r} \end{array} \right],$$

…(12)

where, $R(2\mu) \geq 0, 0 \leq \theta \leq \pi$

VII. PROOF

To prove the result (11), we take

$$F(\theta) = (\sin\theta)^{2\mu} I^{0,N:m_1,n_1;\ldots;m_r,n_r}_{P,Q:p_1,q_1;\ldots;p_r,q_r}$$

$$\left[z_1(\sin\theta)^{2\lambda_1} \ldots z_r(\sin\theta)^{2\lambda_r} \,\middle|\, \begin{array}{l} \left(a_j;\alpha_j^{(1)}\ldots\alpha_j^{(r)};A_j\right)_{1,P}; \\ \left(b_j;\beta_j^{(1)}\ldots\beta_j^{(r)};B_j\right)_{1,Q}; \end{array} \right.$$

$$\left. \begin{array}{l} \left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{1,p_1}\ldots\left(c_j^{(r)},\gamma_j^{(r};C_j^{(r)}\right)_{1,p_r} \\ \left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{1,q_1}\ldots\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r} \end{array} \right]$$

$$= \sum_{v=0}^{\infty} A_v \sin(2v+1)\theta$$

…(13)

Multiplying by sin(2n + 1)θ in both sides of Eq.(13) and integrating between limits 0 to π with respect to θ, we get,

$$F(\theta) = \int_0^{\pi} (\sin\theta)^{2\mu} \sin(2+1)\theta$$
$$I^{0,N:m_1,n_1;\ldots;m_r,n_r}_{P,Q:p_1,q_1;\ldots;p_r,q_r} d\theta . z_1(\sin\theta)^{2\lambda_1} \ldots z_r(\sin\theta)^{2\lambda_r}$$
$$\left[\left| \begin{array}{ll} \left(a_j;\alpha_j^{(1)}\ldots\alpha_j^{(r)};A_j\right)_{1,P}; & \left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{1,p_1}\ldots\left(c_j^{(r)},\gamma_j^{(r};C_j^{(r)}\right)_{1,p_r} \\ \left(b_j;\beta_j^{(1)}\ldots\beta_j^{(r)};B_j\right)_{1,Q}; & \left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{1,q_1}\ldots\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r} \end{array} \right. \right]$$

$$= \sum_{v=0}^{\infty} A_v \int_0^{\pi} \sin(2n+1)\theta \sin(2v+1)\theta d\theta$$

…(14)

$$A_v = \sum_{v=0}^{\infty} \frac{2(-1)^v}{\sqrt{\pi}} . I^{0,N+2:m_1,n_1;\ldots;m_r,n_r}_{P+2,Q+2:p_1,q_1;\ldots;p_r,q_r}$$

$$\left[z_1\ldots z_r \,\middle|\, \begin{array}{l} \left(-\mu+1/2;\lambda_j^i,\ldots,\lambda_j^r\right);\left(-\mu;\lambda_j^i,\ldots,\lambda_j^r\right); \\ \left(b_j;\beta_j^{(1)}\ldots\beta_j^{(r)};B_j\right)_{1,Q+2}; \end{array} \right.$$

$$\left. \begin{array}{l} \left(a_j;\alpha_j^{(1)}\ldots\alpha_j^{(r)};A_j\right)_{1,P+2};\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{1,p_1}\ldots\left(c_j^{(r)},\gamma_j^{(r};C_j^{(r)}\right)_{1,p_r} \\ \left(-\mu\pm\frac{1}{2}\pm v;\lambda_j^i,\ldots,\lambda_j^r\right);\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{1,q_1}\ldots\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r} \end{array} \right],$$

...(15)

From Eq. (14) and Eq.(15), we find the result (11). To prove the result (12), let

$$F(\theta) = \int_0^{\pi} (\sin\theta)^{2\mu} \sin(2n+1)\theta$$
$$I^{0,N:m_1,n_1;\ldots;m_r,n_r}_{P,Q:p_1,q_1;\ldots;p_r,q_r} \left[z_1(\sin\theta)^{2\lambda_1} \ldots z_r(\sin\theta)^{2\lambda_r} \,\middle|\, \begin{array}{l} \left(a_j;\alpha_j^{(1)}\ldots\alpha_j^{(r)};A_j\right)_{1,P}; \\ \left(b_j;\beta_j^{(1)}\ldots\beta_j^{(r)};B_j\right)_{1,Q}; \end{array} \right.$$

$$\left. \begin{array}{l} \left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{1,p_1}\ldots\left(c_j^{(r)},\gamma_j^{(r};C_j^{(r)}\right)_{1,p_r} \\ \left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{1,q_1}\ldots\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r} \end{array} \right] d\theta,$$

$$= \frac{B_0}{2} + \sum_{v=1}^{\infty} B_v \cos v\,\theta \quad , \qquad \ldots(16)$$

Multiplying by cosnθ in both side of the Eq.(16) and integrating between limits from 0 to π,with respect to θ we will get

$$F(\theta) = \int_0^\pi (\sin\theta)^{2\mu} \cos n\theta \, I^{0,N:m_1,n_1;\ldots;m_r,n_r}_{P,Q:p_1,q_1;\ldots;p_r,q_r}$$

$$\left[z_1(\sin\theta)^{2\lambda_1} \ldots z_r(\sin\theta)^{2\lambda_r} \left| \begin{matrix} \left(a_j ; \alpha_j^{(1)} \ldots \alpha_j^{(r)} ; A_j \right)_{1,P} ; \\ \left(b_j ; \beta_j^{(1)} \ldots \beta_j^{(r)} ; B_j \right)_{1,Q} ; \end{matrix} \right. \right.$$

$$\left. \begin{matrix} \left(c_j^{(1)}, \gamma_j^{(1)} ; C_j^{(1)} \right)_{1,p_1} \cdots \left(c_j^{(r)}, \gamma_j^{(r} ; C_j^{(r)} \right)_{1,p_r} \\ \left(d_j^{(1)}, \delta_j^{(1)} ; D_j^{(1)} \right)_{1,q_1} \cdots \left(d_j^{(r)}, \delta_j^{(r)} ; D_j^{(r)} \right)_{1,q_r} \end{matrix} \right] d\theta$$

$$= \int_0^\pi \cos n\theta \left[\frac{B_0}{2} + \sum_{\upsilon=1}^{\infty} B_\upsilon \cos \upsilon\theta \right] d\theta, \qquad \ldots(17)$$

Now using the orthogonality property of trigonometric sine function and Eq.(10),we have

$$B_\upsilon = \frac{2(-1)^\upsilon}{\sqrt{\pi}} \cdot I^{0,N+2:m_1,n_1;\ldots;m_r,n_r}_{P+2,Q+2:p_1,q_1;\ldots;p_r,q_r}$$

$$\left[z_1 \ldots z_r \left| \begin{matrix} (-\mu + 1/2 ; \lambda_j^i, \ldots, \lambda_j^r) ; (-\mu ; \lambda_j^i, \ldots, \lambda_j^r) ; \\ \left(b_j ; \beta_j^{(1)} \ldots \beta_j^{(r)} ; B_j \right)_{1,Q+2} ; (-\mu \pm \upsilon ; \lambda_j^i, \ldots, \lambda_j^r) ; \end{matrix} \right. \right.$$

$$\left. \begin{matrix} \left(a_j ; \alpha_j^{(1)} \ldots \alpha_j^{(r)} ; A_j \right)_{1,P+2} ; \left(c_j^{(1)}, \gamma_j^{(1)} ; C_j^{(1)} \right)_{1,p_1} \ldots \left(c_j^{(r)}, \gamma_j^{(r} ; C_j^{(r)} \right)_{1,p_r} \\ \left(d_j^{(1)}, \delta_j^{(1)} ; D_j^{(1)} \right)_{1,q_1} \ldots \left(d_j^{(r)}, \delta_j^{(r)} ; D_j^{(r)} \right)_{1,q_r} \end{matrix} \right],$$

$$\ldots(18)$$

Form the Eqs. (17) and (18), the result (12) is obtained.

VIII. PARTICULAR CASES

If all the exponent A_j (j = 1, ..., P), B_j (j = 1, ...,Q), $C_j^{(i)}$ (j = 1, ..., pi, i = 1, ..., r) and $D_j^{(i)}$ (j = 1, ..., q_i, i = 1, ..., r) in eq.(11) and (12) are equal to unity, I-function converts in H-function of r variables defined by Srivastava and Panda [2],we get the results are obtained by N Bhati and R K Gupta [10].

IX. CONCLUSIONS

This article focuses on evaluating two integrals that involve the I-function in multiple complex variables, ultimately resulting in the formulation of two separate Fourier series corresponding to its generalized form. The outcomes presented are notably comprehensive, extending and generalizing the earlier results established by R.K. Gupta and N. Bhatti.

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