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Iterative Technique to Find Positive Solutions of Quadratic Fractional Integral Equations

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***Abstract*— In this paper we prove the existence as well as approximations of the positive solutions for a nonlinear quadratic fractional integral equation. An algorithm for the solutions is developed and it is shown that the sequence of successive approximations converges monotonically to the positive solution of related quadratic fractional integral equation under some suitable mixed hybrid conditions. We rely our results on Dhage iteration method embodied in a recent hybrid fixed point theorem of Dhage (2014) for the product of operators in a partially ordered normed linear algebra. An example is also provided to illustrate the abstract theory developed in the paper.**

***Keywords*—Quadratic fractional integral equation; approximate positive solution; Dhage iteration method; hybrid fixed point theorem.**

I. Introduction

The quadratic fractional integral equations have been a topic of interest since long time because of their occurrence in the problems of some natural and physical processes of theuniverse. See Argyros [1], Darwish [2], Darwish and Ntouyas [3], Kilbas et.al. [13], Podlubny [14] and the references therein. The study gained momentum after the formulation of fixed point principles in Banach algebras due to Dhage [4, 5, 6, 7]. The existence results for such equations are generally proved under the mixed Lipschitz and compactness type conditions together with a certain growth condition on the nonlinearities of the quadratic integral equations. See Dhage [5, 6, 7] and the references therein. The Lipschitz and compactness hypotheses are considered to be very strong conditions in the theory of nonlinear differential and integral equations which do not yield any algorithm to determine the numerical solutions. Therefore, it is of interest to relax or weaken these conditions in the existence and approximation theory of quadratic integral equations. This is the main motivation of the present paper. In this paper we prove the existence as well as approximations of the positive solutions of a certain quadratic fractional integral equation via an algorithm based on successive approximations under partially Lipschitz and compactness conditions.

Given a closed and bounded interval $J = [0, T]$; of the real line R, T > 0, we consider the quadratic fractional integral equation (in short QFIE)

$$x(t) = [f(t,x(t))]\left(q(t) + \frac{1}{\Gamma_q}\int_0^t (t-s)^{q-1} g(s,x(s))ds\right), \quad t \in J \tag{1.1}$$

where $f; g : J \times R \to R$ and $q : J \to R$are continuous functions, $1 \leq q < 2$ and Γ is the Euler gamma function.

By a solution of the QFIE (1.1) we mean a function $x \in C(J, R)$ that satisfies the equation (1.1) on J, where $C(J, R)$ is the space of continuous real-valued functions defined on J.

$$x(t) = f(t, x(t)), t \in J \tag{1.2}$$

and if $f(t, x) = 1$ for all $t \in J$ and $x \in R$, it is reduced to nonlinear usual fractional Volterra integral equation

$$x(t) = q(t) + \frac{1}{\Gamma_q}\int_0^t (t-s)^{q-1} g(s,x(s))ds, \quad t \in J \tag{1.3}$$

Therefore, the QFIE (1.1) is general and the results of this paper include the existence and approximations results for above nonlinear functional and Volterra integral equations as special cases.The paper is organized as follows: In the following section we give the preliminaries and auxiliary results needed in the subsequent part of the paper. The main result is included in Section 3. In Section 4 some concluding remarks are presented.

II. Auxiliary Results

Unless otherwise mentioned, throughout this paper that follows, let E denote a partially ordered real normed linear space with an order relation $\preccurlyeq$ and the norm $\|\cdot\|$.

It is known that E is regular if $\{x_n\}_{n\in N}$ is a non decreasing (resp. nonincreasing) sequence in E suchthat $x_n \to x^*$ as $n \to \infty$, then $x_n \preccurlyeq x^*$ (resp. $x_n \succcurlyeq x^*$) for all $n \in N$. Clearly, the partially ordered Banach space C(J;R) is regular and the conditions guaranteeing the regularity of any partially ordered normed linear space E may be found in Heikkila and Lakshmikantham [12] and the references therein.

We need the following definitions in the sequel.

Definition 2.1. A mapping $T: E \to E$ is called **isotone** or **non decreasing** if it preserves the order relation $\preccurlyeq$, that is, if $x \preccurlyeq y$ implies $Tx \preccurlyeq Ty$ for all $x, y \in E$.

Definition 2.2 (Dhage [8]). A mapping $T: E \to E$ is called **partially continuous** ata point $a \in E$ if for $\epsilon > 0$ there exists $\delta > 0$ such that $\|Tx - Ta\| < \epsilon$ whenever xis comparable to a and $\|x - a\| < \delta$. T is called partially continuous on E if it is partially continuous at every point of it. It is clear that if T is partially continuous on E, then it is continuous on every chain C contained in E.

Definition 2.3. A mapping $T: E \to E$ is called **partially bounded** if T (C) is bounded for every chain C in E. T is called **uniformly partially bounded** if all chains T (C)in E are bounded by a unique constant. T is called bounded if T (E) is a bounded subset of E.

Definition 2.4. A mapping $T: E \to E$ is called **partially compact** if T (C) is a relatively compact subset of E for all totally ordered sets or chains C in E. T is called uniformly partially compact if T (C) is a uniformly partially bounded and partially compact on E. T is called partially totally bounded if for any totally ordered and bounded subset C of E, T (C) is a relatively compact subset of E. If T is partially continuous and partially totally bounded, then it is called **partially completely continuous** on E.

Definition 2.5 (Dhage [8]). The order relation $\preccurlyeq$ and the metric d on a non-empty set E are said to be **compatible** if $\{x_n\}_{n\in N}$ is a monotone, that is, monotone non decreasing or monotone non increasing sequence in E and if a subsequence $\{x_{n_k}\}_{n\in N}$ of $\{x_n\}_{n\in N}$ converges to x^* implies that the whole sequence $\{x_n\}_{n\in N}$ converges to x^*.

Similarly, givena partially ordered normed linear space $(E, \preccurlyeq, \|\cdot\|)$, the order relation $\preccurlyeq$ and the norm $\|\cdot\|$ are said to be compatible if $\preccurlyeq$ and the metric d defined through the norm $\|\cdot\|$ are compatible.

Clearly, the set R of real numbers with usual order relation $\leq$ and the norm defined by the absolute value function $|\cdot|$ has this property. Similarly, the finite dimensional Euclidean space R^n with usual component wise order relation and the standard norm possesses thecompatibility property.

Definition 2.6 (Dhage [6]) A upper semi-continuous and non decreasing $\psi: R_+ \longrightarrow R_+$ is called a D-function provided $\psi(0) = 0$. Let $(E, \preccurlyeq, \|\cdot\|)$ be a partially ordered normed linear space. A mapping $T: E \to E$ is called **partially nonlinear D-Lipschitz** if there exists a D-function $\psi: R_+ \longrightarrow R_+$ such that

$$\|Tx - Ta\| \leq \psi(\|x - y\|) \tag{2.1}$$

for all comparable elements $x, y \in E$.

If $\psi(r) = kr$, $k > 0$ then T is called a partially Lipschitz with a Lipschitz constant k.

Let $(E, \preccurlyeq, \|\cdot\|)$ be a partially ordered normed linear algebra. Denote

$$E^+ = \{x \in E | x \succcurlyeq \theta \text{ ,where θis the zero element of E}\}$$

and

$$K = \{E^+ \subset E |\, uv \in E^+ for\ all\ u, v \in E^+\}. \tag{2.2}$$

The elements of the set K are called the positive vectors in E. Then following lemma follows immediately from the definition of the set K and which is often times used in the applications of hybrid fixed point theory in Banach algebras.

Lemma 2.7 (Dhage [7]). If $u_1, u_2, v_1, v_2 \in K$ are such that $u_1 \preccurlyeq v_1 and\ u_2 \preccurlyeq v_2\ then$

$u_1 v_1 \preccurlyeq u_2 v_2.$

Definition 2.8. An operator $T: E \to E$ is said to be positive if the range R(T) of T is such that $R(T) \subseteq K$.

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The Dhage iteration method is embodied in the following hybrid fixed point theorem proved in Dhage [9] which is a useful tool in what follows. A few other hybrid fixed point theorems involving the Dhage iteration method may be found in Dhage [10].

Theorem 2.9. Let $(E, \preccurlyeq, \|\cdot\|)$ be a regular partially ordered complete normed linear algebra such that the order relation $\preccurlyeq$ and the norm $\|\cdot\|$ in E are compatible in every compact chain of E. Let $A, B: E \to K$ be two non decreasing operators such that

(a) A is partially bounded and partially nonlinear D-Lipschitz with D-function ψ_A,

(b) B is partially continuous and uniformly partially compact, and

(c) $M\psi_A(r) < r\ , r > 0,$

where $M = sup\{\|B(C)\|: C\ is\ a\ \text{chain in E}\}$, and

(d) there exists an element $x_0 \in X such that x_0 \preccurlyeq Ax_0 Bx_0 or x_0 \succcurlyeq Ax_0 Bx_0$

Then the operator equation

$$AxBx = x \qquad (2.3)$$

has a positive solution x^* in E and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = A\ x_n B\ x_n$, n = 0; 1;……. converges monotonically to x^*.

Remark 2.10. The compatibility of the order relation $\preccurlyeq$ and the norm $\|\cdot\|$ in every compact chain of E holds if every partially compact subset of E possesses the compatibility property with respect to $\preccurlyeq$ and $\|\cdot\|$.

III. Main Result

The QFIE (1.1) is considered in the function space C(J;R) of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\leq$ in C(J;R) by

$$\|x\| = sup|x(t)| \quad t \in J$$

And
$$x \leq y \Leftrightarrow x(t) \leq y(t) \qquad (2.3)$$

For all $t \in J$ respectively. Clearly, C(J;R) is a Banach algebra with respect to above supremum norm and is also partially ordered w.r.t. the above partially order relation $\leq$.

It is known that the partially ordered Banach algebra C(J;R) has some nice properties w.r.t. the above order relation in it. The following lemma follows by an application of Arzella-Ascolli theorem.

Lemma 3.1. Let $(C(J,R), \leq, \|\cdot\|)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\leq$ defined by (3.1) and (3.2) respectively. Then $\|\cdot\|$ and $\leq$ are compatible in every partially compact subset of C(J;R).

Proof. Let S be a partially compact subset of C(J;R) and let $\{x_n\}_{n\in N}$ be a monotone non decreasing sequence of points in S. Then we have

$$x_1(t) \leq x_2(t) \leq \cdots \leq x_n(t) \leq \cdots, \qquad (3.1)$$

for each $t \in R_+$.

Suppose that a subsequence $\{x_{n_k}\}_{n\in N}$ of $\{x_n\}_{n\in N}$ is convergent and converges to a point x in S. Then the subsequence $\{x_{n_k}(t)\}_{n\in N}$ of the monotone real sequence $\{x_n(t)\}_{n\in N}$ is convergent. By monotone characterization, the whole sequence $\{x_n(t)\}_{n\in N}$ is convergent and converges to a point x(t) in R for each $t \in R_+$.. This shows that the sequence $\{x_n(t)\}_{n\in N}$ converges point-wise in S. To show the convergence is uniform, it is enough to show that the sequence $\{x_n(t)\}_{n\in N}$ is equicontinuous. Since S is partially compact, everychain or totally ordered set and consequently $\{x_n\}_{n\in N}$ is an equicontinuous sequence by Arzela-Ascoli theorem. Hence $\{x_n\}_{n\in N}$ is convergent and converges uniformly to x. As a result $\|\cdot\|$ and $\leq$ are compatible in S. This completes the proof.

We need the following definition in what follows.

Definition 3.2. A function $u \in C(\text{J}, \text{R})$ is said to be a lower solution of the QFIE (1.1) if it satisfies

$$u(t) \leq [f(t, u(t))]\left(q(t) + \frac{1}{\Gamma_q}\int_0^t (t-s)^{q-1} g(s, u(s))ds\right), \quad for\ all\ t \in J. \qquad (3.1)$$

Similarly, a function $v \in C(\text{J}, \text{R})$ is said to be a lower solution of the QFIE (1.1) if it satisfies the above inequalities with reverse sign.

We consider the following set of assumptions in what follows:

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(A_0) q defines a continuous function $q: J \longrightarrow R_+$.

(A_1) defines a continuous function $f: J \times R \longrightarrow R_+$.

(A_2) There exists a real number $M_f > 0$ such that $f(t,x) \le M_f$ $for\ all\ t \in J\ and x \in R$.

(A_3) There exists a D function ϕ, such that

$0 \le f(x,y) - f(t,y) \le \phi(x-y), for\ all\ t \in J\ and x, y \in R, x < y.$

(B_1) g defines a function $g: J \times R \longrightarrow R_+$.

(B_2) There exists a real number $M_g > 0$ such that $g(t,x) \le M_g$ $for\ all\ t \in J\ and x \in R$.

(B_3) $g(t,x)$ $is\ nondecreasing\ in\ x\ for\ all\ t \in J$.

(B_4) $The\ QFIE$ (1,1) has a lower solution $u \in C(J,R)$.

Theorem 3.3. Assume that hypotheses $(A_0) - (A_3)$ and $(B_1) - (B_4)$ hold. Furthermore, assume that

$$\left(\|q\| + \frac{M_q T^q}{\Gamma_q}\right)\phi(r) < r\ , r > 0, \qquad (3.3)$$

Then the QFIE (1.1) has a positive solution x^* defined on J and the sequence $\{x_n\}_{n \in N \cup \{0\}}$ of successive approximations defined by

$$x_{n+1}(t) = [f(t, x_n(t))]\left(q(t) + \frac{1}{\Gamma_q}\int_0^t (t-s)^{q-1} g(s, x_n(s))ds\right), \quad t \in J \qquad (3.4)$$

where $x_0 = u$, converges monotonically to x^*.

Proof: Set E = C(J;R). Then, from Lemma 3.1 it follows that every compact chain in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\le$ in E.

Define two operators A and B on E by

$$Ax(t) = f(x, x(t)), \qquad t \in J \qquad (3.5)$$

$$Bx(t) = q(t) + \frac{1}{\Gamma_q}\int_0^t (t-s)^{q-1} g(s, x(s))ds, \ \ t \in J \qquad (3.6)$$

From the continuity of the integral and the hypotheses $(A_0) - (A_1)$ and (B_1), it follows that A and B define the maps $A, B: E \longrightarrow K$. Now by definitions of the operators A and B, the QFIE (1.1) is equivalent to the operator equation

$$Ax(t)Bx(t) = x(t)\ , \quad t \in J. \qquad (3.7)$$

We shall show that the operators A and B satisfy all the conditions of Theorem 2.9. This is achieved in the series of following steps.

Step I: A and B are non decreasing on E.

Let $x, y \in E$ be such that $x \ge y$. Then by hypothesis (A_2), we obtain

$$Ax(t) = f(x, x(t)) \ge f(x, y(t)) = Ax(t)\ \ for\ all\ t \in J$$

This shows that A is non decreasing operator on E into E. Similarly using hypothesis (B_3), it is shown that the operator B is also non decreasing on E into itself. Thus, A and B are non decreasing positive operators on E into itself.

Step II: A is a partially bounded and partially D-Lipschitz on E.

Let $x \in E$ $be\ arbitrary. Then\ by$ (A_2),

$$|Ax(t)| \le |f(x, y(t))| \le M_f, for\ all\ t \in J.$$

Taking supremum over t, we obtain $\|Ax\| \le M_f$ and so, A is bounded. This further implies that A is partially bounded on E.

Next, let $x, y \in E$ be such that $x \ge y$. Then by hypothesis (A_3),

$$|Ax(t) - Ay(t)| \le |f(t,x(t)) - f(t,y(t))| \le \phi(|x(t) - y(t)|) \le \phi(\|x - y\|), for\ all\ t \in J$$

Taking supremum over t, we obtain

$$\|Ax - Ay\| \le \phi(\|x - y\|), for\ all\ x, y \in E\ \ with x \ge y$$

Hence A is a partially nonlinear D-Lipschitz on E which further implies that A is a partially continuous on E.

Step III: B is a partially continuous on E.

Let $\{x_n\}_{n \in N}$ be a sequence in a chain C of E such that $x_n \to x$ for all $n \in N$. Then, by dominated convergence theorem, we have

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$$\lim_{n\to\infty} Bx_n(t) = \lim_{n\to\infty} q(t) + \lim_{n\to\infty} \frac{1}{\Gamma_q}\int_0^t (t-s)^{q-1} g(s, x_n(s))ds,$$

$$= q(t) + \frac{1}{\Gamma_q}\int_0^t (t-s)^{q-1}[\lim_{n\to\infty} g(s, x_n(s))]ds$$

$$= q(t) + \frac{1}{\Gamma_q}\int_0^t (t-s)^{q-1} g(s, x(s))ds = Bx(t), \text{for all } t \in J$$

This shows that Bx_n converges monotonically to Bx pointwise on J.

Next, we will show that$\{Bx_n\}_{n\in N}$ is an equicontinuous sequence of functions in E.

Let $t_1, t_2 \in J$ with $t_1 < t_2$ Then

$$|Bx_n(t_1) - Bx_n(t_2)| \le |q(t_1) - q(t_2)| + \frac{1}{\Gamma_q}\left|\int_0^{t_2} (t_2-s)^{q-1} g(s,x(s))ds - \int_0^{t_1} (t_1-s)^{q-1} g(s,x(s))ds\right|$$

$$\le |q(t_1) - q(t_2)| + \frac{1}{\Gamma_q}\left|\int_0^{t_2} (t_2-s)^{q-1} g(s,x(s))ds - \int_0^{t_2} (t_1-s)^{q-1} g(s,x(s))ds\right|$$
$$+\frac{1}{\Gamma_q}\left|\int_0^{t_2} (t_1-s)^{q-1} g(s,x(s))ds - \int_0^{t_1} (t_1-s)^{q-1} g(s,x(s))ds\right|$$

$$\le |q(t_1) - q(t_2)| + \frac{1}{\Gamma_q}\left|\int_0^{t_2} |(t_2-s)^{q-1} - (t_1-s)^{q-1}|\,|g(s,x(s))|ds\right|$$
$$+\frac{1}{\Gamma_q}\left|\int_{t_1}^{t_2} (t_1-s)^{q-1} |g(s,x(s))|ds\right|$$
$$\le |q(t_1) - q(t_2)| + \frac{M_q}{\Gamma_q}\left|\int_0^{T} |(t_2-s)^{q-1} - (t_1-s)^{q-1}|ds\right| + \frac{M_q T^{q-1}}{\Gamma_q}|t_1 - t_2|$$
$$\to 0 \quad \text{as} \quad t_2 - t_1 \to 0$$

uniformly for all $n \in N$. This shows that the convergence $Bx_n \to$Bx is uniform and hence B is a partially continuous on E.

Step IV: B is a uniformly partially compact operator on E.

Let C be an arbitrary chain in E. We show that B(C) is a uniformly bounded andequicontinuous set in E. First we show that B(C) is uniformly bounded. Let y ∈ B(C) beany element. Then there is an element x ∈ C be such that y = Bx. Now, by hypothesis(B2),

$$|y(t)| \le |q(t)| + \frac{1}{\Gamma_q}\int_0^{t_2} (t-s)^{q-1} |g(s, x(s))|ds$$

$$\le \|q\| + \frac{M_q T^q}{\Gamma_q} \le r \ \textit{for all } t \in J.$$

Taking supremum over t, we obtain $\|y\| = \|Bx\| \le r$ for all y ∈B(C). Hence,B(C) is a uniformly bounded subset of E. Moreover, $\|B(C)\| \le r$ for all chains C in E.Hence, B is a uniformly partially bounded operator on E.

Next, we will show that B(C) is an equicontinuous set in E. Let $t_1, t_2 \in J$ with $t_1 < t_2$Then, for any y ∈B(C), one has

$$|y(t_2) - y(t_1)| \le |Bx(t_2) - Bx(t_1)|$$
$$\le |q(t_1) - q(t_2)| + \frac{1}{\Gamma_q}\left|\int_0^{t_2} (t_2-s)^{q-1} g(s,x(s))ds - \int_0^{t_1} (t_1-s)^{q-1} g(s,x(s))ds\right|$$
$$\le |q(t_1) - q(t_2)| + \frac{1}{\Gamma_q}\left|\int_0^{t_2} (t_2-s)^{q-1} g(s,x(s))ds - \int_0^{t_2} (t_1-s)^{q-1} g(s,x(s))ds\right|$$
$$+\frac{1}{\Gamma_q}\left|\int_0^{t_2} (t_1-s)^{q-1} g(s,x(s))ds - \int_0^{t_1} (t_1-s)^{q-1} g(s,x(s))ds\right|$$

$$\le |q(t_1) - q(t_2)| + \frac{1}{\Gamma_q}\left|\int_0^{t_2} |(t_2-s)^{q-1} - (t_1-s)^{q-1}|\,|g(s,x(s))|ds\right|$$
$$+\frac{1}{\Gamma_q}\left|\int_{t_1}^{t_2} (t_1-s)^{q-1} |g(s,x(s))|ds\right|$$

$$\le |q(t_1) - q(t_2)| + \frac{M_q}{\Gamma_q}\int_0^{T} |(t_2-s)^{q-1} - (t_1-s)^{q-1}|ds + \frac{M_q T^{q-1}}{\Gamma_q}|t_1 - t_2|$$
$$\to 0 \quad \text{as} \quad t_2 - t_1 \to 0$$

Uniformly for all y ∈B(C).Hence B(C) is an equicontinuous subset of E. Now, B(C) is a uniformly bounded and equicontinuous set of functions in E, so it is compact byArzell_a-Ascoli theorem. Consequently, B is a uniformly partially compact operator on Einto itself.

Step V: u satisfies the operator inequality $u \le AuBu$.

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By hypothesis (B_4), the QFIE (1.1) has a lower solution u defined on J. Then, we have

$$u(t) \leq [f(t,u(t))]\left(q(t) + \frac{1}{\Gamma_q}\int_0^t (t-s)^{q-1} g(s,u(s))ds\right), \quad for\ all\ t \in J. \tag{3.8}$$

From definitions of the operators A and B it follows that u(t) ≤ Au(t) Bu(t) $for\ all\ t \in J$. Hence $u \leq AuBu$.

Step VI: D-function ϕ satisfies the growth condition $M\phi(r) < r$, $r > 0$.

Finally, the D-function ϕ of the operator A satisfies the inequality given in hypothesis (d) of Theorem 2.9, viz.

$$M\psi_A(r) \leq \left(\|q\| + \frac{M_q T^q}{\Gamma_q}\right)\phi(r) \leq r \ \ for\ all\ r > 0.$$

Thus A and B satisfy all the conditions of Theorem 2.9 and we apply it to conclude that the operator equation Ax Bx = x has a positive solution. Consequently the integral equation and the QFIE (1.1) has a positive solution x^*defined on J. Furthermore, the sequence $\{x_n\}_{n\in N}$ of successive approximations defined by (3.4) converges monotonically to x^*. This completes the proof.

Example 3.4. Given a closed and bounded interval J = [0; 1], consider the QFIE,

$$x(t) = [2 + tan^{-1}x(t)]\left(\frac{t}{t-1} + \frac{1}{\Gamma_{\frac{3}{2}}}\int_0^t (t-s)^{1/2}\left[\frac{|1+\tanh x(s)|}{4}\right]ds\right), \quad for\ all\ t \in J \tag{3.9}$$

Here, $q(t) = \frac{t}{t-1}$ which is continuous and $\|q\| = 1/2$. Similarly, the functions f and g are defined by

$f(t,x) = 2 + tan^{-1}x and g(t,x) = \frac{|1+\tanh x|}{4}$. The function f satisfies the hypothesis (A_3) with $\phi(r) = \frac{r}{1+\xi^2} for\ 0 < \xi < r$. To see this, we have

$$0 \leq f(t,x) - f(t,y) \leq \frac{1}{1+\xi^2} \cdot (x-y)$$

for all $x, y \in R$, $x \geq y\ and\ x > \xi > y$.

Moreover, the function f(t; x) is bounded on $J \times R$with bound $M_f = 3$ and so the hypothesis (A_2) is satisfied. Again, since g is bounded on$J \times R$ by $M_g = 1/2$, the hypothesis (B_2) holds. Furthermore, g(t; x) is nondecreasing in xfor all $t \in J$, and thus hypothesis (B_3) is satisfied. Also condition (3.3) of Theorem 3.3 isheld. Next, we have

$$\left(\|q\| + \frac{M_q T^q}{\Gamma_q}\right)\phi(r) = \left(\frac{1}{2} + \frac{2}{3}\frac{1}{\sqrt{\pi}}\right)\frac{r}{1+\xi^2} < r for\ 0 < \xi < r.$$

Finally, the QFIE (3.9) has a lower solution u(t) = 0 defined onJ. Thus all hypotheses of Theorem 3.3 are satisfied. Hence we apply Theorem 3.3 andconclude that the QFIE (3.9) has a positive solutionx^*defined on J and the sequence$\{x_n\}_{n\in N}$defined by

$$x_n(t) = [2 + \tan^{-1}x_n(t)]\left(\frac{t}{t-1} + \frac{1}{\Gamma_{\frac{3}{2}}}\int_0^t (t-s)^{1/2}\left[\frac{|1+\tanh x_n(s)|}{4}\right]ds\right) \tag{3.10}$$

for all $t \in J$, where x_0, converges monotonically to x^*.

IV. CONCLUSION

Finally, while concluding this paper we mention that the nonlinear quadratic fractional integral equation considered here is of very simple nature for which we have illustrated the Dhage iteration method to obtain the algorithms for the positive solutions under weaker partially Lipschitz and compactness type conditions. However, an analogous study could also be made for other complex quadratic fractional integral equations those mentioned in Darwish [2], Darwish and Ntouyas [3] and Dhage and Ntouyas [3] using the similarmethod with appropriate modi_cations. Some of the results along this line will be reported elsewhere.

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