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Analytical Study of Weak Convergence in Projective Tensor Product

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Abstract :- **This paper presents the Study of Weak Convergence in Projective Tensor Product.**

Here, we consider some important classes of Bounded Linear Operators including Projective Topology π on Locally Convex Spaces E and F; U & V be the closed absolutely Convex neighborhoods of 0 in E and F respectively, forming the set τ (U $\otimes$ V) = absolutely Convex hull of U $\otimes$ V in E $\otimes$ F. Here, it is proved in this paper that the Projective topology π with the bounded Linear functional on Projective Tensor Product Characterize Weak Convergence.

Keywords-- **Projective Topology, Locally Convex Spaces, Convex hull, Projective Tensor Product, Bounded Linear Operators, Weak Convergence, Bounded Linear Functional.**

I. INTRODUCTION

E.G.EFFROS (1) and Kothe (3,4) are the pioneer workers of the present area. In fact, the present work is the extension of work done by Halub(2) and Tomiyama(11), Prasad et al.(5), Srivastava et al.(6), Srivastava et al.(7), Srivastava et al.(8), Srivastava et al. (9), Srivastava et al.(10), Yadav et al. (12) and Yadav et al.(13). In this paper, we have studied analytically, weak convergence in Projective Tensor Product.

Here, we use the following definitions, Notations and Fundamental Ideas:

Definition 1: Any projection associated with a direct sum decomposition of a projection on a Linear space X is a linear map $P:X \to X$ such that $P^2 = P$

Theorem 1 : Let X be a linear space,

(i) If $P:X \to X$ is a projection then $X = \text{ran } P \oplus \ker P$

(ii) If $X = M \oplus N$ where M and N are Linear subspaces of X then there is a projection

$P:X \to X$ with ran P = M and ker P = N.

Proof:

For (i) We show that $x \in \text{ran } P$ if $x = Px$

If $x = Px$ then clearly $x \in \text{ran } P$

If $x \in \text{ran } P$ then $x = Py$ for some $y \in X$

And since $P^2 = P$ which follows that $Px = P^2y = Py = x$

If $x \in \text{ran } P \cap \ker P$ then $x = Px$ & $Px = 0$

So $\text{ran } P \cap \ker P = \{0\}$. If $x \in X$ then

We have $x = Px + (x - Px)$; where $Px \in \text{ran } P$ and $(x - Px) \in \ker P$.

Since $P(x - Px) = Px - P^2x = Px - Px = 0$

Thus $X = \text{ran } P \oplus \ker P$. ………………………(1.1)

Now for (ii)

We consider if $X = M \oplus N$ then $x \in N$ has unique decomposition $x = y+z$ with $y \in M$ & $Z \in N$ and $Px = y$ defines the required Projection .

In particular, in orthogonal subspaces while using Projective Topology π, let us suppose that M is a closed subspace of Projective Topology π then by well known property we have $\pi = M \oplus M^{\perp}$. We call the projection of π on to M along $M^{\perp}$ the orthogonal projection of π on to M.

If $x = y+z$ and $x_1 = y_1 + z_1$ where $y, y_1 \in M$ and $z, z_1 \in M^{\perp}$ then by orthogonality of M and $M^{\perp} \Rightarrow \langle Px, x_1\rangle = \langle y, y_1 + z_1\rangle = \langle y, y_1\rangle$

$= \langle y+z, y_1 \rangle$

$= \langle x, Px_1 \rangle$ …………… (1.2)

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Which states that an orthogonal projection is self Adjoint. We show the properties (1.1) and (1.2) characterize orthogonal projections with Defn-1.

Proposition : (a) A Linear functional on a Complex Projective Topology π is a Linear map from π to C. A Linear functional φ is bounded or continuous, if there exists a constant M such that $|\varphi(x)| \leq M \| x \|$ for all $x \in \pi$.

The norm of bounded linear functional φ is

$$\| \varphi \| = \sup_{\| x \| = 1} |\varphi (x)|$$

If $y \in \pi$ then $\varphi_y (x) = < y, x >$ is a bounded Linear functional on π, with

$\| \varphi_y \| = \| y \|$.

(b) If φ is a bounded Linear functional on a Projective Topology π , then there is a unique vector $y \in \pi$ such that

$$\varphi (x) = < y, x > \qquad \text{for all } x \in \pi$$

Definition - 2 : Let E and F be locally convex spaces, and let U & V be the closed absolutely convex neighbourhoods of O in E and F respectively, forming the set $\Gamma (U \otimes V)$ = absolutely convex hull of $U \otimes V$ in $E \otimes F$, ($E \otimes F$ is denoted as tensorial product of E & F).

Definition - 3 : If {U} and {V} are neighbourhood bases in E and F respectively with U, V closed absolutely convex, then the family $\{\Gamma(U \otimes V)\}$ is a neighbourhood basis of a locally convex topology on $E \otimes F$.

This topology is called the projective topology on $E \otimes F$ and is denoted as $E \otimes_{\pi} F$.

Proposition (c): Let $p(x)$ and $q(y)$ be the semi-norms defined by U and V respectively. The set $\Gamma (U \otimes V)$ is absorbing and thus defines a semi-norm. The semi-norm of $\Gamma (U \otimes V)$ is given by

$$p \otimes q (z) = \inf \sum_{i=1}^{n} p(x_i)\, q(y_i)$$

where the infimum is taken over all representations

$$z = \sum x_i \otimes y_i \quad \text{in } E \otimes F .$$

Proof : First we show $\Gamma (U \otimes V)$ is absorbing. Let

$z = \sum_{i=1}^{n} x_i \otimes y_i$ be an element of $E \otimes F$. We observe that

$$\frac{x_i}{p(x_i)} ✞ U \quad \text{if } p(x_i) \neq 0$$

and $$\frac{y_i}{q(y_i)} ✞ V \quad \text{if } q(y_i) \neq 0$$

also $p(x_k) = 0$ iff $\rho . x_k$ ✞ U all $\rho > 0$ and

$q(y_j) = 0$ iff $\rho . y_j$ ✞ V all $\rho > 0$. So we may write

$$z = \sum_{i=1}^{n} x_i \otimes y_i$$

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$$= \sum_i p(x_i)\, q(y_i) \left[\frac{x_i}{p(x_i)} \otimes \frac{y_i}{q(y_i)}\right]$$

$$+ \ \delta \sum_k q(y_k) \left[\frac{x_k}{\delta} \otimes \frac{y_k}{q(y_k)}\right]$$

$$+ \ \delta \sum_j p(x_j) \left[\frac{x_j}{p(x_j)} \otimes \frac{y_j}{\delta}\right]$$

$$+ \ \delta^2 \sum_m \left[\frac{x_m}{\delta} \otimes \frac{y_m}{\delta}\right]$$

In each of the four terms in the sum representing Ƶ, the quantity in the brackets [ ] is in Γ $(U \otimes V)$. Given $\in > 0$, we choose δ sufficiently small so that

$$(*):- Ƶ \in \left(\sum_{i=1}^{n} p(x_i)\, q(y_i) + \in \right) \Gamma\, (U \otimes V).$$

So Γ $(U \otimes V)$ is absorbing.

Now Γ $(U \otimes V)$ is absorbing convex also, so it defines a semi-norm $r(Z-)$ on $E \otimes F$. We now show $r(Z-) = p \otimes q(Z-)$.

(i) r(Ƶ) $\subseteq$ $p \otimes q$(Ƶ) , r(Ƶ) is defined by

r(Ƶ) = inf λ, Ƶ $\in \lambda\, \Gamma\, (U \otimes V)$. By (*) above

r(Ƶ) $\leq$ inf $\Sigma\, p(x_i)\, q(y_i) + \in$ = $p \otimes q$(Ƶ) + $\in$, $\in$ arbitrary yields r(Ƶ) $\leq$ $p \otimes q$(Ƶ).

(ii) $p \otimes q$(Ƶ) $\leq$ r(Ƶ), suppose Ƶ $\in$ $\lambda\, \Gamma\, (U \otimes V)$. Then Ƶ = $\Sigma\, \alpha_k (x'_k \otimes y'_k)$ with $p(x'_k) \leq 1$, $q(y'_k) \leq 1$, $\Sigma\, |\alpha_k| \leq \lambda$ and $\alpha_k \geq 0$. For this particular representation of Ƶ, we see

$\Sigma\, p\, (\alpha_k\, x'_k)\, q(y'_k)$ = $\Sigma\, |\alpha_k| \leq \lambda$. So,

$p \otimes q$(Ƶ) = inf $\Sigma\, p(x_i)\, q\, (y_i) \leq \lambda$.

This is true for every λ with Ƶ $\in \lambda\Gamma(U \otimes V)$.

Thus $p \otimes q$(Ƶ) $\leq$ inf λ, Ƶ $\in \lambda\Gamma(U \otimes V)$

= r(Ƶ).

This completes the proof.

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Proposition (d) : The projective tensor product $E \underset{\pi}{\otimes} F$ of two normed space E, p and F, q is a normed space with norm $p \otimes q$.

If E and F are metrizable locally convex spaces with semi-norms $p_1 \leq p_2 \leq ....$ and $q_1 \leq q_2 \leq ....$ respectively, then $E \underset{\pi}{\otimes} F$ is metrizable with defining semi-norms $p_1 \otimes q_1 \leq p_2 \otimes q_2 \leq .... .$

Proof : Follows immediately from preposition (c) and definition (3).

Thus, from above definitions, theorem, propositions (a), (b), (c) & (d). Which shows the main result as follows:-

Main result :- " If (x_n) is a sequence in Projective Topology π and D is a dense subset of π .

Then (x_n) converges weakly to x if and only if :

(a) II x_n II ≤ M for some constant M ;

(b) < x_n ,y > → < x, y > as n → ∞ for all y ∈ D".

Proof of the Main Result . Suppose that (x_n) is a weakly convergent sequence . We define the bounded linear functional $\varphi_n(x) = < x_n , x >$. Then II φ_n II = II x_n II . Since ($\varphi_n(x)$) converges for each x ∈ π , it is a bounded sequence , and the uniform boundedness theorem implies that {‖ φ_n ‖} is bounded . It follows that a weakly convergent sequence satisfies (a) & Part (b) is trival.

Conversely, suppose that (x_n) satisfies (a) and (b), if z ∈ π, then for any ∈ > 0 there is a y ∈ D such that ‖z - y‖ < ∈, and there is an N such that

I< x_n – x, y >I < ∈ for n≥N . Since ‖ x_n ‖ ≤ M , it follows from the Cauchy-Schwarz inequality that for n ≥ N

$$I< x_n - x, z >I \leq I< x_n - x, y >I + I< x_n - x, z - y >I$$

$$\leq \in + \| x_n - x \| \| z - y \|$$

$$\leq (I + M + \| x \|)\in.$$

Thus , < x_n – x , z > → 0 as n → ∞ for every z∈ π , so x_n → x.

Hence proved.

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REFERENCES

[1] E.G. EFFROS and E.C. LANCE. (1970): Tensor Products of operator algebras.

[2] Halub, J.R. (1970): Tensor Product Mapping Math. Ann., Vol.188, pp 01-12.

[3] Kothe, G. (1969): Topological Vector Spaces, Springer - Verlage, I, New York.

[4] Kothe, G. (1979): Topological Vector Spaces, Springer - Verlag, II, New York.

[5] Prasad Binod, Shahabuddin MD, Pandey, P.K., Shukla S.K. & Srivastava, U.K. (2023): Journal of Mathematics and statistical Science, U.S.A., Vol. - 9(4), pp. 11-18.

[6] Srivastava, U.K., Kumar, S., and Singh, T.N. (2011): J.P.A.S. (Mathematical Science), Vol.17, pp 188-191.

[7] Srivastava, U.K., Kumar, N., Kumar, S., and Singh, T.N. (2012): J.P.A.S. (Mathematical Science), Vol.18, pp189-191.

[8] Srivastava, U.K., and Singh, T.N. (2014): International Journal of Emerging Technology and Advanced Engineering, Vol.4 (3), pp.658-659.

[9] Srivastava, U.K., and Singh, T.N. (2014): International Journal of Emerging Technology and Advanced Engineering, Vol.4 (3), pp.658-659.

[10] Srivastava, U.K.; Pandey, P.K.; & Shrivastava, P.K. (2023): Proceedings of IMBIC, Vol. 12, pp. 150-157.

[11] Tomiyama, J. (1971): Tensor Products and Projection of Norms. One in VON-NEUMANN Algebras.

[12] Yadav, S.K.; Thakur, K.B.; & Srivastava, U.K. (2024) : IJRDET vol. 13(6), pp. 49 – 51.

[13] Yadav, S.K.; Shrivastava, P.K.; Thakur, K.B.; & Srivastava, U.K. (2025): IJSPR Vol. 83, issue 228(02), pp. 5-8.