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A Study on the Existence of Fuzzy Fixed Point Theorems for Some Fuzzy Contractions in Fuzzy Metric Spaces.

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***Abstract* -- In this research paper we tried to study fuzzy fixed point results for fuzzy mappings under some fuzzy contraction conditions in the setting of a complete fuzzy metric space.**

***Keywords*-- Fuzzy set, fuzzy- mapping, fuzzy-fixed-point, fuzzy metric space, Housdorff fuzzy metric.**

I. INTRODUCTION

The concept of fuzzy sets was initiated by Zadeh [1] in 1965. This ground breaking concept challenged the traditional binary logic of true and false by introducing the nation of partial membership, which sets that elements and belong to a set to varying degrees. Fuzzy set theory introduced a new realm of possibilities in various fields, including artificial intelligence, decision-making, control system and pattern recognition.

Numerous examples emerged highlighting how uncertainty in systems often processes a fuzziness district from unpredictable randomness to capture the nuance, non-stationary fuzzy systems, modeled by fuzzy processes were proposed as a natural extension of traditional fuzzy systems in the time domain.

Comprehensive analysis across various mathematical and scientific disciplines has shed light on the distinctive capacity of fuzzy processes to model and handle non stationary uncertainty in dynamical systems.

The best conditions for approximating the solutions of linear and non linear operator equations are provided by fixed point results in the study of mathematical analysis.

Mathematics has traditionally been developed on the basis of precision, exactness and certainty classical logic and classical set theory operate on binary principles, where statements are either true or false and an element either belongs to a set or does not belong to it. While this frame work has been highly successful in pure mathematics and many applied sciences, it often fails to represent the complexities of real world phenomena. In many particular situations, information is incomplete, imprecise or vague and sharp boundaries can not be defined accurately.

II. PRELIMINARIES

In this section fundamental concepts are presented with the goal of providing a thorough understanding to the readers of the basic definitions, examples and lemmas required for a good understanding of our presented results.

III. DEFINITION

3.1 Continuous t-norm :

A Binary Operation

$$* : [0, 1] \times [0, 1] \rightarrow [0, 1]$$

(i) 0 is commutative and associative

(ii) 0 is continuous

(iii) $a * 1 = a \quad \forall \; a \in [0, 1]$

(iv) $a * b \leq c * d \;\; When \; ever \; a \leq c \; and \; b \leq d.$

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3.2 Fuzzy Metric Space :

Let X be a non empty set and * be a continuous t – norm. A mapping

$$N : X \times X \times (0, \infty) \rightarrow [0, 1]$$

(i) $N(x, y, t) > 0$

(ii) $N(x, y, t) = 1 \quad iff \ x = y$

(iii) $N(x, y, t) = N(y, x, t)$

(iv) $N(x, z, t+s) \leq N(x, y, t) * N(y, z, s)$

(v) $N(x, y, \bullet)$ is continuous on $(0, \infty)$

Then $(X, N, *)$ is called a fuzzy metric space.

Definition 3.3

In the fuzzy theory, the fuzzy sets p of the universal set U are defined by a function

$\mu_p : U \rightarrow [0, 1]$ called the membership function of p, where

$$\mu_p(\rho) = 0 \quad if \ \rho \ is \ not \ in \ p$$

$$\mu_p(\rho) = 1 \quad if \ \rho \ is \ totally \ in \ p$$

$$0 < \mu_p(\rho) < 1 \quad if \ \rho \ is \ partially \ in \ p$$

The β - cut of fuzzy sets, p is denoted by $[p]_\beta$ and is defined as

$$[p]_\beta = \{\rho \in U : p(\rho) \geq \beta\}$$

Where $\beta \in [0, 1]$

The set of all fuzzy sets in U will be denoted by F (U).

Definition 3.4

A function

$$* : [0, 1] \times [0, 1] \rightarrow [0, 1]$$

is known as a continuous triangular norm (t- norm) if the following conditions are fulfilled for all $\phi, \mu, \xi, \upsilon \in [0, 1]$

(i) Symmetry : $\phi * \mu = \mu * \phi$

(ii) Monotonicity : $\phi * \mu \leq \xi * \upsilon \quad if \quad \phi \leq \xi \quad and \quad \mu \leq \upsilon$

(iii) Associativity : $(\phi * (\mu * \xi)) = ((\phi * \mu) * \xi)$

(iv) Boundary Conditions : $1 * \phi = \phi$

there basic t - norms are as follows

(i) Lukasiewize t- norm : $(\phi *_L \xi = Max\{\phi + \xi - 1, 0\})$

(ii) Product t- norm : $\phi *_p \xi = \phi \xi$

(iii) Minimum t- norm : that is $\phi *_M \xi = Min\{\phi, \xi\}$

Definition 3.5:

A triplet $\{U, l, *\}$ is a fuzzy metric space if U is a non empty set and *l*, a fuzzy set on $U \times U \times [0, \infty)$ satisfies the following conditions

$$\forall\, \theta, \psi, \omega \in U \quad and \quad \mu, t \geq 0$$

(i) $l(\theta, \psi, t) > 0$

(ii) $l(\theta, \psi, t) = iff \quad \theta = \psi$

(iii) $l(\theta, \psi, t) = l(\psi, \theta, t)$

(iv) $l(\theta, \psi, t) * l(\psi, \omega, \mu) \leq l(\theta, \omega, t + \mu)$

(v) $l(\theta, \psi, \cdot) : [0, \infty) \rightarrow [0, 1]$ is continuous.

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Example 1:

Consider a complete metric space (U, d).

Define $l : U^2 \times [0, \infty) \rightarrow [0, 1]$ as $l(\sigma, \eta, t) = \dfrac{1}{t + d(\sigma, \eta)}$

for $\sigma, \eta \in U$ *and* $t > 0$ and

define $\sigma * \eta = \sigma\eta$ $[or\ \sigma * \eta = \min\{\eta, \sigma\}]$

$\forall \sigma, \eta \in [0, 1]$ then $(U, l, *)$ is a fuzzy metric space on U.

Example 2:

Consider a complete metric space (U, d). and a continuous increasing function $g : R^+ \rightarrow [0, \infty)$

Define $l : U^2 \times [0, \infty) \rightarrow [0, 1]$ as

$$l(\sigma, \eta, t) = e\left(\frac{-d(\sigma, \eta)}{g(t)}\right) \quad \forall \sigma, \eta \in U$$

and t >0, then $(U, l, *)$ is a fuzzy metric space on U with the product t – norm *.

Definition 3.6:

Let $(U, l, *)$ be a fuzzy metric space. A sequence (σ_n) in U is said to be convergent to a point $r \in U$ if $lim_{n\rightarrow\infty} l(\sigma_n, r, t) = 1 \quad \forall\, t > 0.$

Definition 3.7:

A sequence $\{\sigma_\eta\}$ in the fuzzy metric space $(Z, \Lambda, *)$ is called a Cauchy sequence if

$$\Lambda(\sigma_n, \sigma_{n+m}, t) \rightarrow 1 \text{ as } n \rightarrow \infty \text{ for every } m \in N \quad and \quad t > 0.$$

Definition 3.8:

A fuzzy metric space is said to be complete iff Cauchy sequence in it is convergent.

Definition 3.9:

A fuzzy metric space in which every sequence has a convergent sub sequence is said to be compact.

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Definition 3.10:

Let $\{U, l, *\}$ be a fuzzy metric space. The Hausdorff fuzzy metric

$H_l : \mu_l(U) \times \mu_l(U) \rightarrow R$ is defined as

$$H_l(A, B, t) = \min\{\inf \sigma \in A\ (Sup_{\eta \in B}\, l(\sigma,\eta,t)),\ \inf \eta \in B\ (Sup_{\sigma \in A}\, l(\sigma,\eta,t))\}$$

Where $A, B \in \mu_l(U)$ *and* $t > 0.$

$$l(\sigma, A, t) = Sup\ \{l\ (\sigma,\rho,t) :\ \rho \in A\}.$$

Definition 3.11:

Let Z be a metric space and U be any non empty set.

A mapping, $f : U \rightarrow F(Z)$ is called a fuzzy-mapping for convenience, we denote the α-cut set of $f(\sigma)$ by $[\ f\sigma\ ]_\alpha$ instead of $[\ f(\sigma)]_\alpha$.

Definition 3.12:

Any point $r \in U$ is called a fuzzy- fixed point of

$f : U \rightarrow F(U)$ if there is $\alpha \in (0, 1)$ $s.t.\ r \in [f_r]_\alpha$.

Definition 3.13:

A mapping $f : U \rightarrow R$ is called upper semi-continuous if for any sequence of $\{\sigma_n\} \subset U$ and $\sigma \in U,\ \{\sigma_n\} \rightarrow \sigma,\ f(\sigma) \geq \lim\ Sup_{n\rightarrow\infty} f(\sigma_n)$ is implied.

Definition 3.14:

A mapping $f : U \rightarrow R$ is known as lower semi-continuous if, for any $\{\sigma_n\} \subset U$ and $\sigma \in U,\ \{\sigma_n\} \rightarrow \sigma,\ f(\sigma) \leq \lim\ inf_{n\rightarrow\infty} f(\sigma_n)$ is implied.

Definition 3.15:

Consider that $\omega = \{\xi : [0,1] \rightarrow [0,1]\}$ is a collection of all continuous functions such that $\xi(1) = 1,\ \ \xi(0) = 0$ and $\xi(i) > i$ *for all* $0 < i < 1$.

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IV. Lemma

Lemma 4.1

Let $(U, l, *)$ be a fuzzy metric space for all $\sigma, \eta, \in U, l(\sigma,\eta, t)$ is a non decreasing function.

Lemma 4.2

Let $(U, l, *)$ be a fuzzy metric space. Then for each $\rho \in U$ and $\beta \in K_l(U)$ and for $t > 0$ there exists $\eta_0 \in \beta$ such that $l(\rho, \eta_0, t) = l(\rho, \beta, t)$.

Lemma 4.3

Let $(U, l, *)$ be a complete fuzzy metric space $s.t. (K_l, (U), H_l, *)$ is a Housdroff fuzzy metric space on $K_l(U)$. Then for all $\mu, \lambda \in K_l(U)$ *for* $i \in \mu$ and $t > 0$ there exists $J_i \in \lambda$ which satisfies

$$l(i, \lambda, t) = l(i, J_i, t)$$

Then $H_l(\mu, \lambda, t) \le l(i, J_i, t)$

Lemma 4.4

Let $(U, l, *)$ be a complete fuzzy metric space. if there exists $a \in (0, 1)$ $s.t.$ $l(\rho, \eta, at) \ge (\rho, \eta, a)$ for all $\rho, \eta \in U$ and $t \in [0, \infty)$ then $\rho = \eta.$

V. Main Result

Consider a complete fuzzy metric space $(U, l, *)$ and a fuzzy mapping $T: U \to F(U)$ $s.t.$ $\lfloor T_\eta \rfloor_{\alpha T(\eta)} \in K_l(U)$. if there exists α constant, $K \in (0, 1)$ for any $\sigma \in U$ there is $\eta \in K_p^\rho$ satisfying

$$l(\eta, \lfloor T_\eta \rfloor_{\alpha T(\eta)}, Kt) \ge l(\rho,\eta,t) \qquad \text{.......(1)}$$

For t >0 Assume that $(U, l, *)$ satisfies $\lim_{n\to\infty} *_{1=n}^{\infty} l(\rho, \eta, th^i) = 1$ for same ρ_0 in U. Then T has a fuzzy fixed point provided K < P and f is upper semi-continuous.

Proof :

We know that $[T\rho]\alpha T(\rho) \in Kl(U)$ using Lemma 2 and that K_p^ρ is non-empty for any $\rho \in U$ and $p \in (0,1)$. Let $\rho \in U$ be arbitrary, there exits $\rho_1 \in K_p^{\rho_0}$ satisfying $l(\rho_1, [T\rho_1]\alpha T(\rho_1,) Kt) \ge l(\rho_0, \rho_1, t)$ and for $\rho_1 \in U$ r f there exists is $\rho_2 \in K_p^{\rho_1}$ satisfying

$$l\left(\rho_2,[T\rho_2]\alpha T(\rho_2), Kt\right) \geq l(\rho_1, \rho_2, t)$$

Consequently, we obtain a sequence $\{\rho_n\}$ in U $s.t.$ $\rho_{n+1} \in K_p^{\rho_n}$ satisfying

$$l\left(\rho_{n+1},[T\rho_{n+1}]\alpha T(\rho_{n+1}), Kt\right) \geq l(\rho_n, \rho_{n+1}, t) \qquad \text{........(ii)}$$

on the other hand $\rho_{n+1} \in K_p^{\rho_n}$. gives

$$l(\rho_n, \rho_{n+1}, t) \geq l\left(\rho_n,[T\rho_n]\alpha T(\rho_n), pt\right) \qquad \text{.......(iii)}$$

from (ii) and (iii) we obtain

$$l\left(\rho_{n+1},[T\rho_{n+1}]\alpha T(\rho_{n+1}), Kt\right) \geq l\left(\rho_n,[T\rho_n]\alpha T(\rho_n) pt\right).$$

$$l\left(\rho_{n+1},[T\rho_{n+1}]\alpha T(\rho_{n+1}), t\right) \geq l\left(\rho_n,[T\rho_n]\alpha T(\rho_n)\frac{p}{K}t\right). \qquad \text{.......(iv)}$$

Consider that $\gamma = K/p$ then from (iv) we have

$$l\left(\rho_{n+1},[T\rho_{n+1}]\alpha T(\rho_{n+1}), t\right) \geq l\left(\rho_n,[T\rho_n]\alpha T(\rho_n), \frac{t}{\gamma}\right).$$

Now we have

$$l(\rho_n, \rho_{n+1}, t) \geq l\left(\rho_{n-1}, \rho_n, \frac{t}{\gamma}\right) \geq \left(\rho_{n-2}, \rho_{n-1}, \frac{t}{\gamma\cdot\gamma}\right) \geq \ldots$$

$$\ldots\ldots\ldots \geq \left(\rho_0, \rho_1, \frac{t}{\gamma^n}\right) \qquad \text{.......(v)}$$

for all $n \in N$ and $0 < \gamma < 1.$

we choose the constant $s<1$ $s.\,t.\ \gamma s<1$ then $\frac{1}{s}<1$ and $\sum_{i=1}^{\infty}\frac{1}{S^i}<1$ that is $\sum_{i=0}^{m-1}\frac{1}{S^i}<1$ and we have

$$t\left[\frac{1}{S^n}+\frac{1}{S^{n+1}}+\dots\dots\dots\dots\frac{1}{S^{m-2}}+\frac{1}{S^{m-1}}\right]<t,$$

for all $m>n$ and $h>1$. Also we have

$$l(\rho_n,\rho_m,t)\geq l\left(\rho_n,\rho_m,t\left[\frac{1}{S^n}+\frac{1}{S^{n+1}}+\dots\dots\dots\dots\frac{1}{S^{m-2}}+\frac{1}{S^{m-1}}\right]\right)$$

$$\geq l\left(\rho_n,\rho_{n+1},{}^{t}\!/\!_{s^n}\right)*\ l\left(\rho_{n+1},\rho_{n+2},{}^{t}\!/\!_{s^{n+1}}\right)*\dots\dots\dots*l\left(\rho_{m-1},\rho_m,{}^{t}\!/\!_{s^{m-1}}\right)$$

This implies that

$$l(\rho_n,\rho_m,t)\geq\left[l\left(\rho_0,\rho_1,{}^{t}\!/\!_{(\gamma s)^n}\right)*l\left(\rho_0,\rho_1,{}^{t}\!/\!_{(\gamma s)^{n+1}}\right)*\dots\dots l\left(\rho_0,\rho_1,{}^{t}\!/\!_{(\gamma s)^{m-1}}\right)\right]$$

$$l(\rho_n,\rho_m,t)\geq *\ i_{i=n}^{i=\infty}\left[l\left(\rho_0,\rho_1,{}^{t}\!/\!_{(\gamma s)^i}\right)\right]\qquad\dots\dots\text{(vi)}$$

Then

$$\underset{m,n\to\infty}{Limit}\ l(\rho_n,\rho_m,t)=1$$

Thus $\{\rho_n\}$ is a Cauchy sequence in U. Since U is complete, there exists $q\in U$ $s.\,t.\ \underset{n\to\infty}{Lim}\ \rho_n=q$

[$\because$ $(U,l,*)$ be fuzzy metric space satisfying $\underset{n\to\infty}{Lim} *_{i-1}^{i=\infty} l(\rho.\,\eta.th^i)=1$ for $\rho,\eta,\in U,\ t>0\ and\ h>1.$ Suppose that $\{\rho_n\}$ is a sequence in U that stratifies

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$l(\rho_n, \rho_{n+1}, Kt) \geq l(\rho_{n-1}, \rho_n, t)$ $\forall\, n \in N$ and $0 < K < t$ Then $\{\rho_n\}$ is Cauchy sequence.]

from (ii) and (iii) it is clear that

$f(\rho_n) = l\,(\rho_n [T\rho_n] \alpha T(\rho_n), t)$ is increasing and hence converges to 1. Since f is upper semi-continuous.

We have

$$1 = \lim_{n\to\infty} Sup\ f(\rho_n) \leq f\,(q) \leq 1.$$

This implies that $f\,(q) = 1$

so $l\ \ (q, [T\,q\,] \alpha T(q\,), t) = 1$

$$\Rightarrow q \in [T\,q\,] \alpha T(q\,).$$

Thus q is a fuzzy fixed point of T.

VI. Conclusion

In this paper we have investigated the existence and uniqueness of fixed points for certain classes of fuzzy contraction mappings in fuzzy metric spaces. The framework of fuzzy metric space introduced by Kramosil and Michalek (1975) and later modified by George and Veeramani (1994) has been used and the foundational structure for the present study.

We established sufficient conditions under which a fuzzy contraction mapping admits a unique fixed point in a complete fuzzy metric space.

Fixed-point theory acknowledged for its versatility in numerous mathematical realms, provides crucial methods for validating both the existence and uniqueness of solutions in our study we investigated the fixed point theorem for fuzzy mapping under various fuzzy contractive conditions in the context of complete fuzzy metric spaces. Moreover, we supported our results with examples and demonstrated their applicability across various contraction methods.

REFERENCES

[1] L.A. Zadeh (1965) – fuzzy sets.
[2] Kramosil & Michalck (1975) fuzzy metric space.
[3] George & Veeramani (1994) on some results in fuzzy metric space.
[4] Azam, A. Kanwal, S. Common fixed point results for multivalued mapping in Housdroff Intuitionistic fuzzy metric space.
[5] Kanwal. S. Ali A. Al Mazrooe : A Santos – Garcia, G. Existence of fuzzy fixed points of set valued fuzzy mapping in metric and fuzzy metric space.
[6] Jafari, S. Research on fuzzy logic and mathematics with applications.
[7] Kramosil, I. Michalck, J. fuzzy metric and statistical metric spaces.
[8] Ashraf, M.S. Ali, R. Hussain, N. New fuzzy – fixed point Results in Generalized fuzzy metric spaces with application to Integral Equations.
[9] Gregori, V.; Minana, J., Roig, B., Sapena, A. On completeness and Fixed Point Theorems in Fuzzy Metric Space.
[10] Kaur, S., Ozkan, A.; Alanazi, F.D. A new perspective on fuzzy mapping theory with invertedly open and closed mapping.