International Journal of Recent Development in Engineering and Technology
Website: www.ijrdet.com (ISSN 2347-6435(Online) Volume 15, Issue 02, February 2026)

Bitopological Harmonious Labeling of Subdivision of Some Graphs

S. Chandrakala[1], M. Subbulakshmi[2], G. Siva Prijith[3*]

[1]*Associate Professor, PG & Research Department of Mathematics, Tirunelveli Dakshina Mara Nadar Sangam College, T. Kallikulam, India*

[2]*Associate Professor, PG and Research Department of Mathematics, G. Venkataswamy Naidu College, Kovilpatti. India*

[3]*Research Scholar, G. Venkataswamy Naidu College, Kovilpatti. Affiliated to Manonmaniam Sundaranar University, India*

***Abstract*— Bitopological Harmonious labeling for a graph $G=(V(G),E(G))$ with n vertices, is an injective function $f:V(G)\to 2^X$, where X is any non – empty set such that $|X|=m,\ m<n$ and $\{f(V(G))\}$ forms a topology on X, that induces an injective function $f^*:E(G)\to 2^{X^*}$, defined by $f^*(uv)=f(u)\cap f(v)$ for every $uv\in E(G)$ such that $\{f^*(E(G))\}$ forms a topology on X^* where $X^*=X\backslash\{1,2,\dots,m\}$. A graph that admits Bitopological Harmonious labeling is called a Bitopological Harmonious graph. In this paper, we discuss Bitopological Harmonious labeling of subdivision of some graphs.**

***Keywords*— Bitopological Harmonious graph, subdivision, star graph, bistar graph, hurdle graph.**

I. INTRODUCTION

In this paper we consider only simple, finite and undirected graphs. The graph G has a vertex set V = V(G) and edge set E = E(G). For notations and terminology we refer to Bondy and Murthy[2]. Acharya[1] established another link between graph theory and point set topology. Selestin Lina S and Asha S defined Bitopological Star labeling for a graph $G=(V,E)$ as X be any non-empty set if there exists an injective function $f:V(G)\to 2^X$ which induces the function $f^*:E(G)\to 2^X$ as $f^*(v_1v_2)=[f(v_1)\cup f(v_2)]^c$ for every $v_1,v_2\in V(G)$, if $\{f(V(G))\}$and $\{f^*(E(G))\}\cup X$ are topolologies on X then G is said to be Bitopological star graph. In this paper we proved subdivision of some graphs are Bitopological Harmonious graph.

II. BASIC DEFINITIONS

Definition 2.1

Bitopological Harmonious labeling of a graph $G=(V(G),E(G))$ with n vertices is an injective function $f:V(G)\to 2^X$, where X is any non – empty set such that $|X|=m,\ m<n$ and $\{f(V(G))\}$ forms a topology on X, that induces an injective function $f^*:E(G)\to 2^{X^*}$, defined by $f^*(uv)=f(u)\cap f(v)$ for every $uv\in E(G)$ such that $\{f^*(E(G))\}$ forms a topology on X^* where $X^*=X\backslash\{1,2,\dots,m\}$. A graph that admits Bitopological Harmonious labeling is called a Bitopological Harmonious graph.

Definition 2.2

A complete bipartite graph $K_{1,n}$ or $K_{n,1}$ is called a star graph.

Definition 2.3

Bistar graph $B_{m,n}$ is obtained from K_2 by attaching m pendent edges to one end of K_2 and n pendent edges to the other end of K_2.

Definition 2.4

A graph obtained from a path P_n by attaching pendent edges to every internal vertices of the path is called hurdle graph. Hurdle graph with $n-2$ hurdle is denoted by Hd_n.

Definition 2.5

If $e = uv$ is an edge of G and u' is not a vertex of G, then e is said to be subdivided when it is replaced by an edge uu' and $u'v$. The graph obtained by subdividing each edge of a graph G is called the subdivision of G and is denoted by $S(G)$.

III. Main Results

Theorem 3.1

The subdivision of star graph $S(K_{1,n}), n \geq 1$ is a Bitopological Harmonious graph.

Proof:

Let $G = S(K_{1,n})$.

Let $V(G) = \{u_i/0 \leq i \leq n\} \cup \{u_i'/1 \leq i \leq n\}$.

Let $E(G) = \{u_0u_i'/1 \leq i \leq n\} \cup \{u_i'u_i/1 \leq i \leq n\}$.

$|V(G)| = 2n + 1, |E(G)| = 2n$.

Let $X = \{1,2, \dots, |V(G)| - 1\}$.

Define a function $f: V(G) \rightarrow 2^X$ *as follows:*

$f(u_1) = \phi$;

$f(u_i) = \{1,2, \dots, 2(i-1)\}$ for $2 \leq i \leq n$;

$f(u_i') = \{1,2, \dots, 2i-1\}$ for $1 \leq i \leq n$;

$f(u_0) = \{1,2, \dots, 2n\}$.

Here all the vertex labels are distinct and they form a topology on X.

Then the induced function $f^*: E(G) \rightarrow X^*$ is given by

$f^*(uv) = f(u) \cap f(v)$ for all $uv \in E(G)$.

$f^*(u_0u_i') = f(u_i')$ for $1 \leq i \leq n$;

$f^*(u_i'u_i) = f(u_i)$ for $1 \leq i \leq n$.

Since f is 1-1 and so f^*. Also $\{f^*(E(G))\}$ forms a topology on X^*.

Hence f is a Bitopological Harmonious labeling and G is a Bitopological Harmonious graph.

Example 3.2

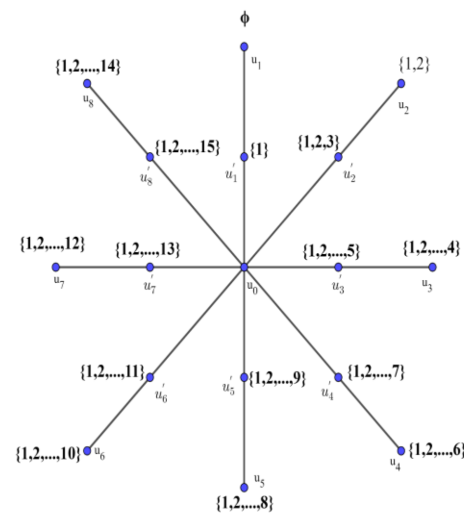

Fig 3.1 Bitopological Harmonious labeling of $S(K_{1,8})$

Theorem 3.3

The subdivision of bistar graph $S(B_{m,n}), m, n \geq 1$ is a Bitopological Harmonious graph.

Proof:

Let $G = S(B_{m,n})$.

$V(G) = \{u, v, w\} \cup \{u_i/1 \leq i \leq m\} \cup \{u_i'/1 \leq i \leq m\} \cup \{v_i/1 \leq i \leq n\} \cup \{v_i'/1 \leq i \leq n\}$.
$E(G) = \{uw\} \cup \{vw\} \cup \{uu_i'/1 \leq i \leq m\} \cup \{u_iu_i'/1 \leq i \leq m\} \cup \{vv_i/1 \leq i \leq n\} \cup \{v_i'v_i/1 \leq i \leq n\}$.

$|V(G)| = 2m + 2n + 3, |E(G)| = 2m + 2n + 2$.

Let $X = \{1,2, \dots, |V(G)| - 1\}$.

Define a function $f: V(G) \rightarrow 2^X$ *as follows:*

$f(u_1) = \phi$;

$f(u_i) = \{1,2, \dots, 2(i-1)\}$ for $2 \leq i \leq m$;

$f(u_i') = \{1,2, \dots, 2(i-1) + 1\}$ for $1 \leq i \leq m$;

$f(v_i) = \{1,2, \dots, 2m + 2i - i\}$ for $1 \leq i \leq n$;

$f(v_i') = \{1,2, \dots, 2m + 2i\}$ for $1 \leq i \leq n$;

$f(u) = \{1,2, \dots, 2m\}$;

International Journal of Recent Development in Engineering and Technology
Website: www.ijrdet.com (ISSN 2347-6435(Online) Volume 15, Issue 02, February 2026)

$f(u) = \{1,2,\ldots,2n+1\}$;

$f(v) = \{1,2,\ldots,2m+2n+2\}$.

Here all the vertex labels are distinct and they form a topology on X.

Then the induced function $f^*: E(G) \rightarrow X^*$ is given by

$f^*(uv) = f(u) \cap f(v)$ for all $uv \in E(G)$.

$f^*(u_i'u_i) = f(u_i)$ for $1 \le i \le m$;

$f^*(uu_i') = f(u_i')$ for $1 \le i \le m$;

$f^*(v_i'v_i) = f(v_i)$ for $1 \le i \le n$;

$f^*(vv_i') = f(v_i')$ for $1 \le i \le n$;

$f^*(uw) = f(u)$;

$f^*(wv) = f(v)$;

Since f is 1-1 and so f^*. Also $\{f^*(E(G))\}$ forms a topology on X^*.

Hence f is a Bitopological Harmonious labeling and G is a Bitopological Harmonious graph.

Example 3.4

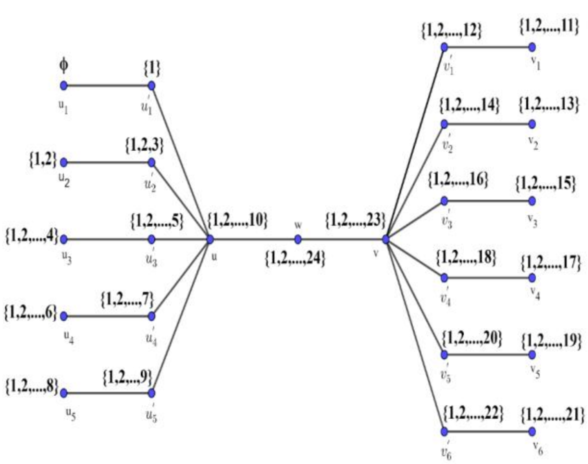

Fig 3.2 Bitopological Harmonious labeling of $S(B_{5,6})$

Theorem 3.5

The subdivision of hurdle graph $S(Hd_n), n \ge 3$ is a Bitopological Harmonious graph.

Proof:

Let $G = S(Hd_n)$.
$V(G) = \{u_i / 1 \le i \le m\} \cup \{u_i' / 1 \le i \le n-1\} \cup \{v_i / 1 \le i \le n-2\} \cup \{v_i' / 1 \le i \le n-2\}$.
$E(G) = \{u_i u_i' / 1 \le i \le n-1\} \cup \{u_i' u_{i+1} / 1 \le i \le n-1\} \cup \{v_i v_i' / 1 \le i \le n\} \cup \{v_i' u_{i+1} / 1 \le i \le n-2\}$.
$|V(G)| = 4n-5, |E(G)| = 4n-6$.

Let $X = \{1,2,\ldots,|V(G)|-1\}$.

Define a function $f: V(G) \rightarrow 2^X$ *as follows:*

$f(v_1) = \phi$;

$f(v_i) = \{1,2,\ldots,i-1\}$ for $2 \le i \le n-2$;

$f(u_i) = \{1,2,\ldots,2(n-2)+2(i-1)\}$ for $1 \le i \le n$;

$f(u_i') = \{1,2,\ldots,2(n-2)+2i-1\}$ for $1 \le i \le n-1$;

$f(v_i') = \{1,2,\ldots,n+i-3\}$ for $1 \le i \le n-2$;

Here all the vertex labels are distinct and they form a topology on X.

Then the induced function $f^*: E(G) \rightarrow 2^{X^*}$ is given by

$f^*(uv) = f(u) \cap f(v)$ for all $uv \in E(G)$.

Here $f^*(u_i'u_i) = f(u_i)$ for $1 \le i \le m$;

$f^*(u_iu_i') = f(u_i)$ for $1 \le i \le n-1$;

$f^*(u_i'u_{i+1}) = f(u_i')$ for $1 \le i \le n-1$;

$f^*(v_iv_i') = f(v_i)$ for $1 \le i \le n$;

$f^*(v_i'u_{i+1}) = f(v_i')$ for $1 \le i \le n-2$.

Since f is 1-1 and so f^*. Also $\{f^*(E(G))\}$ forms a topology on X^*.

Hence f is a Bitopological Harmonious labeling and G is a Bitopological Harmonious graph.

International Journal of Recent Development in Engineering and Technology
Website: www.ijrdet.com (ISSN 2347-6435(Online) Volume 15, Issue 02, February 2026)

Example 3.6

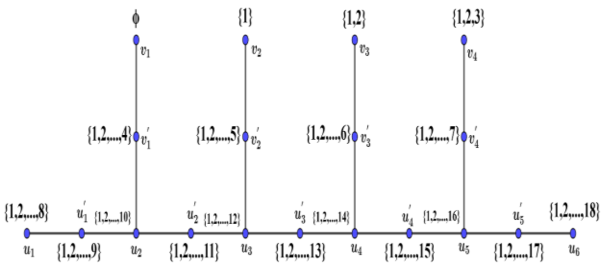

Fig 3.3 Bitopological Harmonious labeling of S(Hd_6)

IV. CONCLUSION

In this paper, we prove that the subdivisions of star, bistar, spider, and hurdle graphs are Bitopological Harmonious graphs. We further extend our work to investigate their applications.

REFERENCES

[1] Acharya B.D., Set valuations and their applications, MRI Lecture note in Applied Mathematics, No.2, Mehta Research Institute of Mathematics and Mathematical Physics, 1983.

[2] Bondy J.A and Murthy U.S.R, "Graph Theory and Application" (North Holland). New York (1976).

[3] Joseph A Gallian 2018, 'A Dynamic Survey of Graph Labeling', The Electronic Journal of Combinatorics.

[4] Selestin Lina S, Asha S, 'On Topological Cordial Graphs', Journal of Science and Technology, 5(2020), 25-28.

[5] Selestin Lina S, Asha S, Topological cordial labeling of some graphs', Malaya Journal of Matematik, Vol. 9, No. 1, 861-863.

[6] Selestin Lina S. & Asha, S. (2022), 'Bitopological labeling of tree related graphs', AIP Conference Proceedings. 2385. 130016. 10.1063/5.0070851.

[7] G. Siva Prijith, M. Subbulakshmi, S. Chandrakala, 'Topological Cordial Labelling of Some Graphs', Mapana – Journal of Sciences 2023, Vol. 22, Special Issue 1, 129-140 ISSN 0975-3303.

[8] S Chandrakala, G Siva Prijith, M Subbulakshmi, 'Bitopological Harmonious Labeling of Some Acyclic Graphs by Using Python Program', Metallurgical and Materials Engineering, Volume 31(6), 2025,p159-170.

[9] M Subbulakshmi, S Chandrakala, G Siva Prijith,Bitopological Harmonious Labeling of Some Star Related Graphs', International Journal of Mathematics Trends and Technology, Volume 71, Issue 7, 2022,pp 44-48, July 2025.

[10] M Subbulakshmi, G Siva Prijith, S Chandrakala, 'Topological Cordial Labeling for Some Different Types of Graphs', International Journal of Food and Nutritional Sciences, Volume 11, Issue 12,2022.