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Introduction to Finsler Geometry

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Abstract-- **In this paper we discussed the concept of geometry, differential geometry and Finsler Geometry.in modern times geometric have been extended they sometimes show a high level of abstraction and complexity. The geometry which deals with the help differential calculus is called differential geometry. In the present era the models of Finsler geometry have much importance in applications. we have discussed here some special Finsler spaces which have much importance applications. Therefore, we give some special Finsler spaces which are based on their metrics, torsion tensors and curvature tensors.**

Keywords-- **Finsler spaces, tensors, geometry.**

I. A Brief Historical Development Of Finsler Geometry

Finsler geometry is a kind of differential geometry, which was originated by P. Finsler [9] in 1918. Main focus of Finsler in his dissertation was to geometrize calculus of variations, the idea given by his teacher Caratheodory. The terms of Finsler geometry were present in the epoch-making lecture of B. Riemann which he delivered in June 1854, at Gottingen University. In the lecture Riemann had discussed various possibilities by means of which an *n*-dimensional space may be equipped with the metric before coming to the square root metric.

$$ds = \sqrt{g_{ij}(x)dx^i dx^j} \qquad (i, j=1,2,3,........, n),$$

where the coefficients g_{ij} are functions of coordinates x^i and $det(g_{ij}) \neq 0.$ This quadratic differential form is called a Riemannian metric and space with such metric is called a Riemannian space.

He had thought over cubic and quartic metrics also, but he gave up them due to the difficulty of assigning geometrical meaning to various differential invariants; furthermore, the computation was very complicated. Consequently, he concluded that the theory of such generalized metrics (cubic and quartic) would hardly contribute to the progress of geometry. We quote the following from the famous lecture of Riemann delivered in 1854:

"Investigations of this more general class would actually require no essential different principles but it would be rather time consuming and throw relatively no light on the study of space, especially since results cannot be expressed geometrically."

Finsler geometry is usually considered as a generalization of the Riemannian geometry in which the space consists of tangent bundles instead of collection of points. Finsler spaces differ from Riemannian spaces with the fact that in the former the metric depends on direction also. Riemann's main attention was on a metric where the distance *ds* between two neighboring points represented by the co-ordinates x^i and $x^i + dx^i$ defined by

There are two approaches of Finsler Space out of which one is considered as Riemannian metric generalization. Finsler Space is a space where metric function is given by

$$ds = L(x^1, x^2, x^3, ... x^n,\ x^1,\ dx^1, dx^2,\ dx^3, ...\ dx^n) = L(x,y), (y = dx).$$

We are concerned with the generalized metric $ds = L(x,y)$ which gives the distance between two points x^i and $x^i + dx^i$. Riemann had also discussed that the positive fourth root of a fourth order differential form $(ds^4 = g_{ijmn}\, dx^i\, dx^j\, dx^m\, dx^n)$ might serve as a metric.

These functions have three properties in common:

i. they are positive definite;
ii. they are homogeneous of first degree in the differentials;
iii. they are convex in the differentials.

International Journal of Recent Development in Engineering and Technology
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It would seem natural, therefore, to introduce a further generalization to the effect that the distance between two neighboring points x^i and $x^i + dx^i$ be defined by some function $L(x^i dx^i)$ i.e. $dS = L(x^i, dx^i)$, where L satisfies above three properties. Riemann asserted that the differential geometry based on such generalized metric would develop in a way similar to the case of Riemannian geometry.

Due to Riemann's comments, mathematicians did not try to study such generalized spaces for more than 60 years. In 1918, a 24 years old German named Paul Finsler [9] tried to study such spaces and submitted his thesis to Gottingen University. His approach of studying this geometry was based on calculus of variations. He put the idea of calculus of variations with special reference to new geometrical background, which was given by his teacher Caratheodory. The history of development of Finsler geometry can be divided into the following four periods:

First period : 1924 - 1933,
Second period : 1934 - 1950,
Third period : 1951 - 1963,
Fourth period : 1963 - till date.

The study of Finsler spaces in India was started around 1960 under the leadership of Prof. R.S. Mishra, Prof. R.N. Sen and Prof. K.S. Amur. Some important Indian mathematicians in this fields are Prof. U.P. Singh, Prof. H.D. Pande, Prof. R.B. Mishra, Prof. M.D. Upadhyay, Prof. R.S. Sinha, Prof. B.B. Sinha, Prof. Ram Hit, Dr. B.N. Prasad, Asstt. Professor P.C. Yadav, Prof. H.S. Shukla, Prof. T.N. Pandey, Prof. P.N. Pandey, Prof. S.C. Rastogi, Prof. C.S. Bagewadi, S.K. Narasimhamurthi, Dr. A.K. Dwivedi and some foreign Finslerians are Prof. Z. Shen, H.S. Park, I.Y. Lee, Alkou Tadashi, P.L. Antonelli, R. Miron, H. Akabar Zadeh etc.

Now, I will discuss some preliminary concepts of Finsler geometry which have been used in the present thesis.

II. Homogeneous Function, Curve, Line-Element & Tangent Bundle

A. Homogeneous function is a function with multiplicative scaling behaviour. If the argument is multiplied by a factor, then the result is multiplied by some power of this factor. More precisely, if $f : TM \to W$ is a function between two vector spaces over a field F, and k is an integer then f is said to be homogeneous of degree k in y if $f(x, cy) = c^k f(x, y)$ for all nonzero $c \,\varepsilon\, F$ and $y \,\varepsilon\, V$.

Let *R* be a region of n-dimensional differentiable manifold M^n which is covered completely by a co-ordinate system, such that any point *P* of *R* is represented by a set of n real independent variables $x^i \ (i = 1, 2, 3, \ldots\ldots\ldots n)$, called the co-ordinates of the point. A transformation of co-ordinates is represented by a set of *n*-equations

(2.1) $$x^{i} = x^{i}(x^1, x^2, \ldots\ldots, x^n), \qquad (i' = 1, 2, \ldots\ldots\ldots n)$$

which shows that the co-ordinates x^i of a point *P* of M^n are represented in the new co-ordinate system by new variables $x^{i'}$. We assume that the functions of (2.1) are at least $x^{i'}$ of class C^2 and

(2.2) $$det\left(\frac{\partial xi'}{\partial xi}\right) \neq 0.$$

A set of points of R whose co-ordinates may be expressed as functions of a single parameter 't' is regarded as a curve of M^n. Thus the equations

(2.3) $$x^i = x^i(t)$$

define a curve C of M^n. If the functions (2.3) are of class C^1, we shall regard the entity whose components are given by

(2.4) $$y^i = \frac{dx^i}{dt}$$

International Journal of Recent Development in Engineering and Technology
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as the tangent vector to *C*. We called the combination (x^i, y^i) a line-element of *C*.

B. Tangent bundle : The tangent bundle [3] of a differentiable manifold M^n is the union of the tangent spaces of M^n, that is $TM = \bigcup_{x \varepsilon M} T_x M$, where $T_x M$ denotes the tangent space to M^n at the point *x*. So, an element of *TM* can be thought of as a pair *(x, y)*, where *x* is a point in M^n and *y* is a tangent vector to M^n at *x*. The set of coordinates $\left(\frac{\partial}{\partial x^i}\right)$ defines a basis of the tangent space.

The infinitesimal distance between two points $P(x^i)$ and $Q(x^i, dx^i)$ of curve (2.3) lying on Manifold M^n is defined by $ds = L(x^i, dx^i) = \sqrt{g_{ij}(x, y) dx^i dx^j}$. The arc PQ becomes tangent at x on Manifold M^n.

III. FINSLER SPACE

Let M^n be *n*-dimensional manifold, *TM* tangent bundle of M^n, $\left(\frac{\partial}{\partial x^i}\right)$ the basis of tangent spaces at (x), and $y = (y^i) = \frac{dx^i}{dt}$. A function $L: TM \to [0, \infty)$ of the line-elements $(x^i y^i)$ defined on M^n is called fundamental function if it satisfies the following three conditions:

(a) The function $L(x^i, y^i)$ is positively homogeneous of degree one in y^i i.e.

3.1 $$L(x^i, ky^i) = kL(x^i, y^i), \qquad k > 0$$

That is, the arc length of curve is independent of the choice of parameter t.

(b) The function $L(x^i, y^i)$ is positive if not all y^i vanish simultaneously, i.e.

3.2 $$L(x^i, y^i) > 0 \text{ with } \sum_i (y^i)^2 \neq 0$$

That is, the distance between two distinct points is positive.

(c) The quadratic form

(3.3) $$\dot{\partial}_i \dot{\partial}_j L^2(x^i, y^i)\xi^i \xi^j = \frac{\partial^2 L^2(x^i, y^i)}{\partial x^i \partial y^j} \xi^i \xi^j$$

is assumed to be positive definite for any variable ξ^i.

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That is, $L(x^i, y^i)$ is a convex function in y^i.

The manifold M^n equipped with the fundamental function L is called a Finsler space [3]. It is denoted by F^n or (M^n, L). Some examples of Finsler spaces are Normed vector spaces, Euclidean spaces, Riemannian spaces, Randers spaces, ...

From Euler's theorem on homogeneous functions, we have

(3.4) $$\dot{\partial}_i L(x,y) y^i = L(x,y)$$

and

(3.5) $$\dot{\partial}_i \dot{\partial}_j L(x,y) y^i = 0$$

We put

(3.6) $$g_{ij}(x,y) = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j L^2(x,y)$$

Using the theory of quadratic forms and the condition (c), we deduce from (3.4) that

(3.7) $$g(x,y) = | g_{ij}(x,y) | > 0$$

for all line-elements (x^i, y^i).

If the function L is of particular form

(3.8) $$L(x^i, dx^i) = \sqrt{g_{ij}(x^k) dx^i dx^j},$$

where the coefficients $g_{ij}(x^k)$ are independent of dx^i, the metric defined by this function is called Riemannian metric and the manifold M^n is called a Riemannian space. Throughout the present thesis, the n-dimensional Finsler space will be denoted by F^n or (M^n, L), whereas n-dimensional Riemannian space will be denoted by R^n.

IV. Physical Motivation

In a perfectly homogenous and isotropic medium, geometry is Euclidean, and shortest paths are straight lines. In an inhomogeneous space, geometry is Riemannian and the shortest paths are geodesics. If a medium is not only inhomogeneous, but also unisotropic one, has innate directional structure, the appropriate geometry is Finslerian[13], [14] and the shortest paths are correspondingly Finsler-geodesics. As a consequence the fundamental metric tensor depends on both position and direction. This is also a natural model for high angular resolution diffusion images.

Finsler geometry has its genesis in integral of the form $\int_a^b L(x,y)\,dt$, where $x = x^i, y = y^i = \frac{dx^i}{dt}$. Let us find out some contexts in which this integral arises.

International Journal of Recent Development in Engineering and Technology
Website: www.ijrdet.com (ISSN 2347-6435(Online) Volume 15, Issue 01, January 2026)

(a) Suppose *x* stands for position, *y* for velocity. Then *L(x, y)* would have the meaning of speed and *t* would play the role of time; in this case the integral $\int_a^b L(x,y)\,dt$ measures distance travelled.

(b) In an unisotropic medium (rays and wave fronts are not orthogonal to each other) the speed of light depends on its direction of travel. At each location *x*, visualize *y* as an arrow that emanates from *x*. We denote the time that light takes to travel from *x* to the top of *y* and call the result *L(x,y)*. The integral $\int_a^b L(x,\, dx)\,dt$ represents total time that light takes to traverse in given path in this medium.

(c) It is well-known that the time taken by man in climbing up and going down on same length of the slope of a mountain are distinct. It means time measures function *L(x (t), y (t))* also depends on direction. This fundamental function *L* together with slope of mountain *TM* (Tangent bundle) is Finsler space.

(d) Cost of transportation function not only depends on distance but also on direction, except some other physical perturbation such as friction, air resistance etc. This function can be regarded as fundamental function of Finsler space.

(e) (Mathematical ecology) Suppose *x* stands for the state of coral reef, and *y* displacement vector from the state x to new state x + dx, then *L(x, dx)* represents the energy one needs in order to develop from the state *x* to the neighbouring state *x* + *dx*. Hence the integral $\int_a^b L(x,\, dx)\,dt$ represents the total energy cost of a given path of evolution.

From above contexts we see that the world is Finslerian and Finsler geometry has wide applications in theory of relativity, control theory, thermodynamics, optics, ecology and mathematical biology.

V. Tangent Space, Indicatrix And Cotangent Spaces

We consider a change of local co-ordinates as represented by the equation (2.1). Along the curve (2.3) referred to an invariant parameter t, the new components of the tangent vector $y^{i'} = \frac{dx^{i'}}{dt}$ are obtained by differentiating the relation.

(5.1) $$x^{i'} = x^{i'}(x^i(t))$$

with respect to t, which gives

(5.2) $$y^{i'} = \frac{\partial x^{i'}}{\partial x^i} y^i$$

or, in terms of differentials,

(5.3) $$dx^{i'} = \frac{\partial x^{i'}}{\partial x^i} dx^i.$$

Here dx^i is interpreted as the components of a displacement in M^n from a point P(x^i) to a point $Q(x^i + dx^i)$. If the point $P(x_i)$ is fixed, i.e. the coefficients $\frac{\partial x^{i'}}{\partial x^i}$ of the transformation (5.3) are fixed, this relation represents a linear transformation of the dx^i onto the dx^i. The same is true for the variables y^i and $y^{i'}$ in the transformation (5.2). Therefore, the n entities of this kind may be taken to define the elements of an n-dimensional linear vector space.

International Journal of Recent Development in Engineering and Technology
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A system of n quantities X^i whose transformation law under (2.1) is equivalent to that of the y^i is called a contravariant vector attached to the point $P(x^i)$ of M^n. Such contravariant vectors constitute the elements of a vector space. The totality of all contravariant vectors attached to $P(x^i)$ of M^n is the *tangent space* denoted by $T_n(P)$ or $T_n(x^i)$

A. Indicatrix :

We consider the function $L(x^i, y^i)$ defined for all line-elements (x^i, y^i) over the region R of M^n. The equation,

$$L(x^i, y^i) = 1, \qquad (x^i \text{ fixed, } y^i \text{ variable})$$

Represents an (n - 1)-dimensional locus in $T_n(P)$, i.e., a hypersurface. This hypersurface plays the role of unit sphere in the geometry of the vector space $T_n(P)$ and is called Indicatrix [28].

B. Cotangent space :

Let M^n be a smooth manifold and let x be a point in M^n. Let T_xM be the tangent space at x. Then cotangent space at x is defined as the dual space of T_xM denoted by T_x^*M or $(T_xM)^*$. Concretely, elements of the cotangent space are linear functional on T_xM. That is, every element $f \in T_x^*M$ is a linear map $f : T_xM \rightarrow R^+$ where R^+ is set of positive real numbers. The elements of T_x^*M are called cotangent vectors.

VI. Pull-Back Tangent Bundle, Non-Linear Connection, Decomposition Of T(TM-0) And T*(TM-0) :

*A. Pull-back tangent bundle (□ * T M) :* Let M^n be an n-dimensional manifold. Suppose T_xM is the tangent space at $x \in M$, and $TM = \bigcup_{x \in M} T_xM$ the tangent bundle of M. Each element of $T\ M$ has the form (x, y), where $x \in M$ and $y \in T_xM$. Let $TM_0 = T\ M|\{0\}$. The natural projection $\pi : TM \rightarrow M$ is given by $\pi(x, y) = x$.

The pull-back tangent bundle $\pi^* TM$ is a vector bundle over TM_0 whose fiber

$$\pi_v{}^* TM \text{ at } v \in T\ \ M_0 \text{ is } T_xM \text{, where } \pi(v) = x.$$

Then $\pi^* TM = \{(x, y, v) | y \in T_xM_0, v \in T_xM\}$. The natural basis for $\pi_v{}^* TM$ is $\{\partial_i|_v = (v, \frac{\partial}{\partial x^i})|_x\}$ for all $i = 1, 2, \ldots \ldots n$.

B. Non-linear connection: A non-linear connection on a manifold M^n is a collection of locally defined 1 - homogeneous function N^i_j on *(TM-0)* satisfying transformation rules

$$\frac{\partial \bar{x}^j}{\partial x^i} \bar{N}^h_j = \frac{\partial \bar{x}^h}{\partial x^j} \bar{N}^j_i - \frac{1}{2} \frac{\partial^2 \bar{x}^h}{\partial x^i \partial x^j} y^j \quad \text{and} \tag{6.1}$$

(6.2) $$N_j^i = \frac{\partial G^i}{\partial y^j}.$$

C. Decomposition of T(TM - 0): The vector spaces span $\{\frac{\partial}{\partial x^i}\Big|_y : i = 1,2,......n\}$ depend on local coordinates. Therefore, we cannot say about $"\frac{\partial}{\partial x^i}"$ direction in *T(TM-0)*. However, when M^n is equipped with a non-linear connection N_j^i, let

(6.3) $$\frac{\delta}{\delta x^i}\Big|_y = \frac{\partial}{\partial x^i}\Big|_y - N_i^k(x,y)\frac{\partial}{\partial y^k}\Big| \in T(TM-0),$$

where $\frac{\delta}{dx^i}\Big|_y = \frac{\partial \bar{x}^r}{\partial x^i}\frac{\delta}{\delta x^r}\Big|_y$. Thus 2n-dimensional vector spaces $T_p(TM-O)$ has n-dimensional subspaces, V_pTM = span $\{\frac{\partial}{\partial y^j}\Big|_y\}$ and

$H_pTM =$ span $\{\frac{\delta}{\delta x^i}\Big|_y\}$ and these are independent of local coordinates. Let us define $VTM = \bigcup_{p\in TM-O} V_pTM$ and $HTM = \bigcup_{p\in TM-O} H_pTM$, then $T(TM-O) = VTM \oplus HTM$. The vectors in *VTM* are called vertical vectors and vectors in *HTM* are called horizontal vectors. The tangent of a geodesic is always a horizontal vector; geodesic spray G(x, y) is horizontal for all $(x,y)\in(TM-O)$.

D. Decomposition of T^* (TM-0): On *TM* the 1-forms dx^i and dy^i satisfying law of transformation

(6.4) $$dx^i\Big|_y = \frac{\partial x^i}{\partial \bar{x}^r}\,\partial \bar{x}^r\Big|_y$$

(6.5) $$dy^i\Big|_y = \frac{\partial x^i}{\partial \bar{x}^r}\,\partial \bar{y}^r\Big|_y + \frac{\partial^2 \bar{x}^i}{\partial \bar{x}^r \partial \bar{x}^s}\bar{y}^r\, d\bar{x}^s\Big|_y.$$

Let $\delta y^i\Big|_y = dy^i\Big|_y + N_i^k(x,y)dx^j\Big|_y$, where $\delta y^i\Big|_y = \frac{\partial x^i}{\partial \bar{x}^r}\delta\,\bar{y}^r\Big|_y$. The 2n-dimensional vector spaces, T^* *(TM-0)* has two n-dimensional subspaces $V_p{}^*TM$ span $\{\delta y^i\Big|_p\}$ and $H_p{}^*TM$ span $\{\delta y^j\Big|_p\}$ and these are independent of local coordinates. Then pointwise $T^*(TM-O) = V^*TM \oplus H^*TM$ co-vectors in V^*TM are called vertical co-vectors and co-vectors in H^*TM are called horizontal co-vectors.

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VII. Metric Tensor And Carton Torsion Tensor

From equation (3.6) we can easily see that the quantities g_{ij} defined by it form the components of a covariant tensor of rank 2; also $g_{ij}(x, y)$ are positively homogeneous of degree zero in y^i and symmetric in their indices. Due to homogeneity condition - (a) of section 1.3 for the function $L\,(x, y)$, we have

(7.1) $$L^2(x,y)=g_{ij}(x,y)y^i y^j$$

By condition - (c) of section 1.3 it follows that inverse of matrix g_{ij} exists. Thus, if g^{ij} denotes the inverse of g_{ij}, then

(7.2) $$g_{ij}(x,y)g^{jk}(x,y)=\delta_i^k,$$

where δ_i^k is well known Kronecker delta. Therefore the tensor whose covariant and contravariant components are $g_{ij}(x, y)$ and $g^{ij}(x, y)$ respectively, is called the metric tensor or the first fundamental of the Finsler space F^n.

Cartan torsion tensor :

Let $x \in M,\ y \in T_xM$ and L be the fundamental function on Manifold M^n. Define $c_y : T_xM \times T_xM \times T_xM \to R$ by $c_y(u,v,w)=c_{ijk}u^i v^i w^k$. The family $c=\{c_{ijk}\}$ for all $y \in T_xM$, is called Cartan torsion. The tensor $C_{ijk}(x,y)$ defined by

(7.3) $$C_{ijk}(x,y)=\frac{1}{2}\dot{\partial}_k g_{ij}=\frac{1}{4}\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k L^2$$

is positively homogeneous of degree -1 in y^i and is symmetric in all their indices. This tensor is called Cartan's C-tensor and satisfies

(7.4) $$C_{ijk}(x,y)y^i=C_{ijk}(x,y)y^j=C_{ijk}(x,y)y^k=0$$

(7.5) $$(\partial_h C_{ijk})y^i=(\partial_h C_{ijk})y^j=(\partial_h C_{ijk})y^k=0$$

VIII. Magnitude Of A Vector. The Notion Of Orthogonality

The metric tensor $g_{ij}(x, y)$ may be used in two different ways, in defining the magnitude of a vector and also the angle between two vectors.

Let X^i be a vector, then the scalar X given by

(8.1) $$X^2=(g_{ij}(x,X)X^i X^j$$

is called the magnitude of this vector.

If Y^i is another vector, then the ratio,

International Journal of Recent Development in Engineering and Technology
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(8.2) $$\cos(X,Y) = \frac{g_{ij}(x^i, x^i) x^i y^j}{L(x^i, x^i) L(x^i, y^i)}$$

is called the 'Minkowskian cosine' corresponding to the (ordered) pair of directions X^i, Y^i (Rund[27]). It is obvious from (8.2) that Minkowskian cosine is non-symmetric in X^i and Y^i.

Let X^i be a vector and Y^i an arbitrary fixed direction, then the scalar

(8.3) $$g_{ij}(x,y) X^i X^j$$

is called the square of magnitude of the vector X^i for the pre-assigned direction Y^i. If Y^i is another vector, then the ratio,

(8.4) $$\cos(X,Y) = \frac{g_{ij}(x,y) X^i Y^j}{\sqrt{g_{ij}(x,y) X^i Y^j} \sqrt{g_{ij}(x,y) X^i Y^j}}$$

is called the cosine of X^i, Y^i for the direction y^i.

It is to be noted that the concepts of magnitude of vector and the cosine between two vectors given by (8.3) and (8.4) respectively stand at each point of the space in a pre-assigned direction y^i which has been called the element of support. Also the cosine given by (8.4) is symmetric in X^i and Y^i (Berwald [4], Synge [31]).

To distinguish between the two magnitudes we call the magnitude given by (8.1) as the Minkowskian magnitude of X^i and that given by (8.3) the magnitude of X^i.

The equations (8.2) and (8.4) are used to define the orthogonality in F^n.

The vector is said to be orthogonal with respect to X^i if

(8.5) $$g_{ij}(x, X) X^i Y^j = 0$$

Thus according to this definition if Y^i is orthogonal with respect to X^i then it is not necessary that X^i is also orthogonal with respect to Y^i.

The vectors X^i and Y^i are said to be orthogonal (for a pre-assigned y^i) if

(8.6) $$g_{ij}(x, y) X^i Y^j = 0$$

This definition of orthogonality is symmetric in X^i and Y^i.

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IX. Connections And Covariant Differentiations

Any quantity in a Finsler space is function of line element *(x, y)*. If *S(x, y)* is a scalar field in a Finsler space then $\frac{\partial s}{\partial x^i}$ are not components of a covariant vector. If we have a non-linear connection $N^i_j(x, y)$, we can obtain the covariant vector field of the components.

$$S_{|i} = \frac{\delta s}{\delta x^i}, \; where \; \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^{j\bullet}}$$

Further, if we have quantities $F^i_{jk}(x, y)$ which obey the transformation rule similar to Christoffel symbols, the covariant derivatives $K^i_{j|k}$ of a Finsler tensor field of (1, 1)-type is defined by

(9.1) $$F^i_{j|k} = \frac{\delta K^i_j}{\delta x^k} + K^r_j K^i_{rk} - K^i_r F^r_{jk}$$

On the other hand, the partial derivatives of components of a tensor field K^i_j with respect to y^k give a new tensor field, but we shall modify them as

(9.2) $$F^i_j|_k = \frac{\partial K^i_j}{\partial y^k} + K^r_j C^i_{rk} - K^i_r C^r_{jk},$$

where $C^i_{jk}(x, y)$ are components of a tensor field of (1, 2)-type. The collection $(F^i_{jk}, N^i_j, C^i_{jk}$ constitute a Finsler connection, and covariant derivatives given by (9.1) and (9.2) are called h- and v-covariant derivatives of K^i_j respectively.

A. Finsler connection: Suppose N^i_j is a non-linear connection on M^n and F^i_{jk}, C^i_{jk} are respectively 0 & -1 degree homogeneous functions in y^i from (TM - 0) to $R^+, \aleph(M)$ the set of vector field on manifold M^n. A Finsler connection is a mapping

$$\nabla(F^i_{jk}, N^i_j, C^i_{jk}) : Tp(TM - 0) \times \aleph(M) \rightarrow T_{\pi(p)}(M), (Y, X) \rightarrow \nabla_Y(X)$$

satisfying the properties

(1) ∇ is linear over *R* in X and Y (but not necessarily in y).

(2) If $f \in C^\infty(M)$ and $y \in (T_x M - 0)$ then in local coordinates.

$$\nabla \frac{\delta}{\delta x^i}\Big|_y \Big(f \frac{\partial}{\partial x^j}\Big|_y\Big) = df\Big(\frac{\partial}{\partial x^i}\Big|_y\Big)\frac{\partial}{\partial x^j}\Big|_x\Big) + fF^m_{ij}(y)\frac{\partial}{\partial x^m}\Big|_x,$$

and $\nabla \frac{\partial}{\partial x^i}\Big|_y \Big(f \frac{\partial}{\partial x^j}\Big|_y\Big) = f\, C^i_{jk}(y)\frac{\partial}{\partial x^m}\Big|_x$.

For all $X \in \aleph(M)$ and ∇ does not depend on the local coordinates.

For any Finsler connection $(F^i_{jk}, N^i_j, C^i_{jk})$ we have five torsion tensors and three curvature tensors : hh, hv and vv-curvatures [Riemannian curvature (f), Berwaldian Curvature (B) and third curvature (Q)] which are given by

(9.3) (h)h-torsion : $T^i_{jk} = F^i_{jk} - F^i_{kj}$

(9.4) (v)v-torsion : $S^i_{jk} = C^i_{jk} - C^i_{kj}$

(9.5) (h)hv-torsion: C^i_{jk} as the vertical connection C^i_{jk}

(9.6) (v)h-torsion : $R^i_{jk} = \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j}$

(9.7) (v)hv-torsion : $P^i_{jk} = \dot{\partial}_k N^i_j - F^i_{kj}$

(9.8) h-curvature :

$$R^i_{hjk} = \frac{\delta F^i_{hj}}{\delta x^k} - \frac{\delta F^i_{hk}}{\delta x^j} + F^m_{hj}F^i_{mk} - F^m_{hj}F^i_{mk} + C^i_{hm}F^m_{jk}$$

(9.9) hv-curvature : $P^i_{hjk} = \dot{\partial}_k F^i_{hj} - C^i_{hk|j} + C^i_{hm}P^m_{jk}$

(9.10) v-curvature : $S^i_{hjk} = \dot{\partial}_k C^i_{hj} - \dot{\partial}_j C^i_{hk} + C^m_{hj}C^i_{mk} - C^m_{hk}C^i_{mj}$

The deflection tensor field D^i_j of a Finsler connection $F\Gamma$ is given by

(9.11) $D^i_j = y^k F^i_{jk} - N^i_j$.

When a Finsler metric is given, various Finsler connections are determined from the metric. The well known examples are Cartan's connection, Rund's connection and Berwald's connection.

International Journal of Recent Development in Engineering and Technology
Website: www.ijrdet.com (ISSN 2347-6435(Online) Volume 15, Issue 01, January 2026)

B. Cartan's Connection: We are concerned with a Finsler space $F^n = (M^n, L)$ which is to be endowed with the Cartan's connection $C\Gamma = (\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, C^i_{jk})$ constructed from the fundamental function $L(x, y)$. According to the theory of Finsler connections due to M. Matsumoto ([17], [18]), the $C\Gamma$ is determined from the axiomatic stand- point as follows :

There exists a unique Finsler connection $F\Gamma = (F^i_{jk}, N^i_j, C^i_{jk})$ which satisfies the following five conditions :

(C_1) $g_{ij|k} = 0$

(C_2) (h)h-torsion : $T^i_{jk} = 0$

(C_3) Deflection tensor field $D^i_j = 0$

(C_4) $g_{ij}\big|_k = 0$

(C_5) (v)v-torsion : $S^i_{jk} = 0$

This connection is called the Cartan's connection and is denoted by

$$C\Gamma = (\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, C^i_{jk}).$$

The last two conditions C_4 and C_5 give

(9.12) $$C^i_{jk} = \frac{1}{2} g^{ih} \frac{\partial g_{jk}}{\partial y^h}$$

This shows that vertical connection of $C\Gamma$ and Cartan's C-tensor are identical

The first three conditions C_1, C_2 and C_3 give

(9.13) $$F^i_{jk} = \Gamma^{*i}_{jk} = \frac{1}{2} g^{ih} \left[\frac{\delta g_{jh}}{\delta x^k} + \frac{\delta g_{kh}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^h}\right]$$

(9.14) $$N^i_j = \Gamma^{*i}_{0k} = \gamma^i_{0k} - 2C^i_{km} G^m,$$

where

(9.15) $$G^i = \frac{1}{2} \gamma^i_{00}$$

and

International Journal of Recent Development in Engineering and Technology
Website: www.ijrdet.com (ISSN 2347-6435(Online) Volume 15, Issue 01, January 2026)

(9.16) $$\gamma^i_{jk} = \frac{1}{2} g^{ih} \left[\frac{\partial g_{jh}}{\partial x^k} + \frac{\partial g_{kh}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^h}\right]$$

is the Christoffel symbol of (M^n, L). Here '0' denotes contraction with y^i.

It is easy to verify from the axiomos C_1, C_3 and equation (9.1), that

(9.17) a) $y^i_{|h} = 0$, b) $L_{|h} = 0$, c) $l^i_{|h} = 0$

where l^i is a unit vector in the direction of element of support y^i, i.e.

$$l^i = \frac{y^i}{L(x,y)}$$

Since C_{ijk} is an indicatory tensor, therefore, from (9.2) we have $y^i\big|_h = \delta^i_h$. Thus in view if (9.1) and condition C_1, we have $L|_i = \dot{\partial}_i L = l_i$, where $l_i = g_{ij} l^j$. It may also be verified that

(9.18) a) $l^i\big|_j = L^{-1} h^i_j$, b) $l_i\big|_j = L^{-1} h_{ij}$, c) $l_{i|j} = 0$

d) $h_{ij|k} = 0$ e) $h_{ij}\big|_k = L^{-1}(l_i h_{jk} + l_j h_{ki})$,

where h_{ij} is the angular metric tensor defined by

(9.19) $$h_{ij} = g_{ij} - l_i l_j$$

and $h^i_j = g^{ik} h_{jk}$

C. Rund's Connection: The Rund's connection of a Finsler space $F^n = (M^n, L)$ is a Finsler connection which is obtained from Cartan's connection $C\Gamma$ by the C-process [18]. The C-process is characterized by expelling the torsion tensor C^i_{jk}. Thus the first two connection coefficients of the Rund's connection $R\Gamma$ are the same as those of the Cartan's connection $C\Gamma$, while the third is equal to zero. Thus the Rund's connection $R\Gamma$ of the Finsler space F^n is given by $R\Gamma$ = $(\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, 0)$. The torsion tensors of $R\Gamma$ are such that

(9.20) $$\begin{cases} T^i_{jk}=0,\ R^i_{jk} = \text{the same as that of } C\Gamma, C^i_{jk}=0, \\ P^i_{jk}=\text{the same as that of } C\Gamma,\ S^i_{jk}=0. \end{cases}$$

The curvature tensors of $R\Gamma$ are as follows :

(9.21) $$\begin{cases} a).\ h-curvature & K: K^i_{hjk}=R^i_{hjk}-C^i_{hr}R^r_{jk} \\ b).\ hv-curvature & F: F^i_{hjk}=P^i_{hjk}+C^i_{hk|j}-C^i_{hr}P^r_{jk} \end{cases}$$

while the v-curvature tensor S^i_{hjk} of $R\Gamma$ vanishes identically. We note that h-covariant differentiations with respect to $C\Gamma$ and $R\Gamma$ coincide with each other. Furthermore C^i_{jk} in (9.21) is the Cartan's C-tensor $C^i_{jk}=g^{ih}C_{jhk}$, which is not the vertical connection of $R\Gamma$ as it vanishes for $R\Gamma$.

The h-curvature K and hv-curvature F of $R\Gamma$ may be given in terms of connection coefficients as

(9.22) $$\begin{cases} a). & K^i_{hjk}=\dfrac{\delta\Gamma^{*i}_{hj}}{\delta x^k}-\dfrac{\delta\Gamma^{*i}_{hk}}{\delta x^j}+\Gamma^{*m}_{hj}\Gamma^{*i}_{mk}-\Gamma^{*m}_{hk}\Gamma^{*i}_{mj} \\ b). & F^i_{hjk}=\dot\partial_j\Gamma^{*i}_{hj}. \end{cases}$$

D. Berwald's Connection: The Berwald's connection of a Finsler space $F^n=(M^n,\ L)$ is a Finsler connection which is obtained from Rund's connection $R\Gamma$ by the P^1-process [18]. The P^1 process is characterized by expelling the torsion tensor P^i_{jk}. The Berwald's connection of Finsler space F^n is denoted by $B\Gamma=(G^i_{jk},\ G^i_j,0)$, where

(9.23) a) $G^i_{jk}=\dot\partial_j G^i_k$ b) $G^i_j=\Gamma^{*i}_{0j}=\dot\partial_j G^i$

The Berwald's connection $B\Gamma$ is uniquely determined from metric function $L(x, y)$ of F^n by the following five axioms :

(B_1) $L_{|i}=0$

(B_2) (h) h-torsion: $T^i_{jk}=0$

(B_3) Deflection : $D^i_j=0$

(B_4) (v) hv-torsion : $P^i_{jk}=0$

International Journal of Recent Development in Engineering and Technology
Website: www.ijrdet.com (ISSN 2347-6435(Online) Volume 15, Issue 01, January 2026)

(B_5) (h) hv-torsion : $C^i_{jk} = 0$

Thus the torsion tensors of $B\Gamma$ are such that

(9.24)
$$\begin{cases} T^i_{jk} = 0, \; R^i_{jk} = \text{the same as that of } R\Gamma, \; C^i_{jk} = 0 \\ P^i_{jk} = \text{the same as that of } R\Gamma, \; S^i_{jk} = 0. \end{cases}$$

The v-connection coefficients G^i_{jk} of $B\Gamma$ are related to those of by

(9.25)
$$G^i_{jk} = \Gamma^{*i}_{jk} + C^i_{jk|0}.$$

The curvature tensors of $B\Gamma$ are as follows

(9.26)
$$\begin{cases} a)\ h-curvature\ H : H^i_{hjk} = K^i_{hjk} + C^i_{hj|0|k} - C^i_{hk|0|j} \\ \qquad\qquad + C^i_{kr|0}\, C^r_{jh|0} - C^i_{jr|0}\, C^i_{jh|0} \\ b)\ hv-curvature\ G : G^i_{hjk} = F^i_{hjk} + \dot{\partial}_k\, C^i_{jh|0} \end{cases}$$

The v-curvature tensor S^i_{hjk} of $B\Gamma$ vanishes identically.

The simpler forms of H^i_{hjk} and G^i_{hjk} of $B\Gamma$ may be given by

(9.27)
$$H^i_{hjk} = \dot{\partial}_h\, R^i_{jk}, \qquad G^i_{hjk} = \dot{\partial}_h\, G^i_{jk}$$

It is to be noted that $B\Gamma$ is neither h-metrical nor v-metrical in general :

$$g_{ij(k)} = -2C_{ijk|0}, \qquad g_{ij.k} = C_{ijk},$$

where h- and v-covariant derivatives with respect to $B\Gamma$ are denoted by ( ) and '.' respectively.

X. GEODESICS AND PATHS IN A FINSLER SPACE

The geodesics of a Finsler space are the curves of minimum or maximum arc-length between any two points of the space. The differential equations of a geodesic in a Finsler space is given by [18]

(10.1)
$$\frac{d^2x^i}{ds^2} + 2G^i\left(x, \frac{dx}{ds}\right) = 0,$$

where is the arc-length of the curve $x^i = x^i(s)$ and

(10.2) $$2G^i = \gamma^i_{jk} y^j y^k \quad \text{or}$$

(10.3) $$2G^i = g^{ir}(y^j \dot{\partial}_r \dot{\partial}_j F - \partial_r F).$$

Here Lagrangian function *L* is defined on TM by

$$F(x,y) = \frac{1}{2} L^2(x,y),$$

where $F : TM \to R$ is the Finsler function.

Let M^n be a manifold with a Finsler connection $F\Gamma = (F^i_{jk}, N^i_j, C^i_{jk})$. A curve C of the tangent bundle *T(M)* over M^n is called an h-path if C is the projection of an integral curve of an h-basic vectors field $B^h(v)$ corresponding to a fixed $v \in V^n$ [18].

(10.4) $$\begin{cases} \dfrac{dy^i}{dt} + N^i_j(x(t), y(t))\dfrac{dx^j}{dt} = 0 \\ \dfrac{d^2x^i}{dt^2} + F^i_{jk}(x(t), y(t))\dfrac{dx^j}{dt}\dfrac{dx^k}{dt} = 0. \end{cases}$$

Geodesic spray :

Geodesic spray G $\in \aleph$ (TM - 0), the set of vector fields on (TM-0), is locally defined as

(10.5) $$G|_y = y^i \frac{\partial}{\partial x^i}|_y - 2G^i(x,y)\frac{\partial}{\partial y^i}|_y$$

Here G does not depend on local coordinates and G^i is defined by (10.3).

XI. Special Finsler Spaces

In Riemannian geometry we have many interesting theorems such that if a Riemannian space is assumed to have special geometrical properties, or to satisfy special tensor equations, or to admit special tensor fields, then the space reduces to one of well-known space forms, for instance, Euclidean space, spheres, topological spheres, projective spaces and so on.

On the other hand, in Finsler geometry we have special Finsler spaces, namely, Riemannian spaces and Minkowskian spaces, but there are various kinds of Riemannian spaces and Minkowskian spaces. As a consequence we have an important problem to classify all the Minkowskian spaces. It is easy to write down concrete forms of fundamental function *L(x, y)* which are interesting as a function, for instance, a Randers metric, Kropina metric, generalized Kropina metric, Matsumoto metric and cubic metric.

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It is essential for the progress of Finsler geometry to find Finsler spaces, which are quite similar to Riemannian spaces, but not Riemannian and Minkowskian spaces, which are analogous to flat spaces, but not flat.

In the present section, we are mainly concerned with special tensor equations satisfied by torsion, curvature and other important tensors. In the following, we give some definitions of special Finsler spaces and their corresponding result.

(A) Riemannian space :

A Finsler space $F^n = \left(M^n, L(x,y)\right)$ is said to be a Riemannian space if its fundamental function *L(x, y)* is written as

$$L^2(x,y) = g_{ij}(x) y^i y^j .$$

Among Finsler spaces, the class of all the Riemannian spaces is characterized by $C_{ijk} = 0$ i.e. vertical connection Γ^v of the Cartan's connection $C\Gamma$ is flat.

(B) Locally Minkowskian space :

A Finsler space $F^n = (M^n, L(x, y)$ is called locally Minkowskian space if there exists a co-ordinate system (x^i) in which *L* is a function of y^i only [18].

A Finsler space is locally Minkowskian if and only if

For $C\Gamma$: $R^h_{ijk} = C^h_{ij|k} = 0$

For $R\Gamma$: $K^h_{ijk} = F^h_{ijk} = 0$

For $B\Gamma$: $H^h_{ijk} = G^h_{ijk} = 0$

(C) Berwald space :

If the connection coefficient G^i_{jk} of the Berwald's connection $B\Gamma$ given by

$$G^i_{jk} = \dot{\partial}_j G^i_k$$

are functions of position alone, the space is called a Berwald space [18].

A Finsler space is Berwald space if and only if

For $C\Gamma$: $C^h_{ij|k} = 0$

For $R\Gamma$: $F^h_{ijk} = 0$

For $B\Gamma$: $G^h_{ijk} = 0$

(D) Landsberg space

A Finsler space is called a Landsberg space [18] if the Berwald connection $B\Gamma$ is h-metrical i.e. $g_{ij(k)} = 0$.

In terms of the Cartan's connection $C\Gamma$, a Landsberg space is characterized by

(a) $P^i_{jk} = 0$, or (b) $P^h_{ijk} = 0$

(E) C-reducible Finsler space:

A Finsler space of dimension n, more than two, is called C-reducible if C_{ijk} is written in the form [18] :

$$C_{ijk} = \frac{1}{n+1}\pi_{(ijk)}(h_{ij}C_k)$$

where $C_i = C_{ijk}g^{jk}$ is the torsion vector h_{ij}, is the angular metric tensor given by $h_{ij} = g_{ij} - l_i l_j$ and $\pi_{(ijk)}$ is the sum of cyclic permutation of *i, j, k*.

(F) Semi C-reducible Finsler space:

A Finsler space of dimension n, more than two, is called semi C-reducible if C_{ijk} is written in the form [18] :

$$C_{ijk} = \frac{p}{n+1}\pi_{(ijk)}(h_{ij}C_k) + \frac{q}{C^2}C_iC_jC_k,$$

where $C^2 = g^{ij}\,C_iC_j$ and $p + q = 1$.

(G) Quasi C-reducible Finsler space :

A Finsler space of dimension n, more than two, is called quasi C-reducible if there exists a symmetric Finsler tensor field A_{ij}, satisfying $A_{i0} = 0$, in trms of which is written in the form [18] :

$$C_{ijk} = \pi_{(ijk)}(A_{ij}\,C_k).$$

(H) P-reducible Finsler space :

A Finsler space of dimension n, more than two, is called P-reducible if (v)hv-torsion tensor P_{ijk} of $C\Gamma$ is written in the form ([12], [22]):

$$P_{ijk} = \frac{1}{n+1}\pi_{(ijk)}(h_{ij}C_{k|0})$$

International Journal of Recent Development in Engineering and Technology
Website: www.ijrdet.com (ISSN 2347-6435(Online) Volume 15, Issue 01, January 2026)

(I) C2-like Finsler space :

A Finsler space if called C2-like Finsler space [23] if

$$C_{ijk} = \frac{1}{C^2} C_i C_j C_k$$

(J) C3-like Finsler space :

A Finsler space is called C3-like Finsler space [25] if

$$C_{ijk} = S_{(ijk)} \{ h_{ij} a_k + C_i C_j b_k \}$$

where a_k and b_k are components of arbitrary covariant vectors such that $a_0 = b_0 = 0$ and $S_{(ijk)}$ denotes the cyclic sum of i, j, k.

(K) S3-like Finsler space :

A Finsler space F^n with fundamental function *L(x, y)* is called S3-like Finsler space [18] if v-curvature tensor S_{hijk} of $C\Gamma$ is written in the form

$$L^2 S_{hijk} = S\{ h_{hj} h_{ik} - h_{hk} h_{ij} \},$$

where *S* is a scalar and function of position alone.

(L) S4-like Finsler space :

A Finsler space F^n is called S4-like Finsler space [25] if v-curvature tensor S_{hijk} of $C\Gamma$ is written in the form

$$L^2 S_{hijk} = h_{hj} M_{ik} + h_{ik} M_{hj} - h_{hk} M_{ij} - h_{ij} M_{hk},$$

where M_{ij} are components of a symmetric covariant tensor of second order and are (-2)p-homogeneous in y^i satisfying $M_{0j} = 0.$

(M) R3-like Finsler space :

A Finsler space of dimension more than three, is called R3-like Finsler space [20] if h-curvature tensor R_{hijk} of $C\Gamma$ is written in the form

$$R_{hijk} = g_{hj} L_{ik} + g_{ik} L_{hj} - g_{hk} L_{ij} - g_{ij} L_{hk}$$

where $L_{ik} = \frac{1}{n-1} (R_{ik} - \frac{r}{2} g_{ik}),$ $\qquad R_{ik} = R^h_{ikh}$ and $r = \frac{1}{n-1} R_{ik} g^{ik}.$

International Journal of Recent Development in Engineering and Technology
Website: www.ijrdet.com (ISSN 2347-6435(Online) Volume 15, Issue 01, January 2026)

(N) Finsler space of scalar curvature :

A Finsler space of scalar curvature *K* is characterized by [18] :

$$R_{i0j} = KL^2 h_{ij},$$

where R_{ijk} are components of (v)h-torson tensor of $C\Gamma$ defined by (9.6).

(O) One-form :

A one-form on a differentiable manifold is a smooth section of the cotangent bundle. It is a smooth mapping of the total space of the tangent bundle of M to R whose restriction to each fiber is a linear functional on the tangent space. Symbolically,

$\beta : TM \rightarrow R, \beta_x = \beta|_{TxM} : T_x M \rightarrow R$ where β_x is linear.

In a local coordinate system, a one-form is a linear combination of the differentials of the coordinates: $\beta_x = b_i dx^i$ where the b_i are smooth functions (Fibers over x). It is an order-1 covariant tensor field.

Examples :

1. The second element of a three-vector is given by the one-form [0, 1, 0[. That is, the second element of [x, y, z] is [0,1,0] . [x,y,z] = y.
2. The mean element of an *n*-vector is given by the one-form [1/*n*, 1/*n*, ..., 1/*n*]. That is, *mean [*1/*n*, 1/*n*, 1/*n]*.*v*

XII. Intrinsic Fields Of Orthonormal Frames

Berwald theory of two-dimensional Finsler space is developed based on the intrinsic field of orthonormal frame which consists of the normalized supporting element l^i and unit vector orthonormal to l^i. Following the idea, Moor introduced in a three-dimensional Finsler space, the intrinsic field of orthonormal frame which consists of the normalized supporting element l^i, the normalized torsion vector $C^i|C$ and a unit vector orthogonal to them and developed a theory of three-dimensional Finsler spaces. Generalizing the Berwald's and Moor's ideas, Miron and Matsumoto ([18], [20], [24]) developed a theory of intrinsic orthonormal frame fields on n-dimensional Finsler space as follows.

Let *L(x, y)* be the fundamental function of an n-dimensional Finsler space and introduce Finsler tensor fields of (0, 2i-1) type, i = 1, 2, ...,n by

$$L_{i_1 i_2 \ldots\ldots i_{2\alpha-1}} = \frac{1}{2^\alpha} \dot{\partial}_{i_1} \dot{\partial}_{i_2} \ldots \dot{\partial}_{i_{2\alpha-1}} L^2 .$$

Then we get a sequence of covariant vectors

$$L_{\alpha)i} = L_{ij_1 j_2 \ldots j_{2\alpha-3} \cdot j_{2\alpha-2}} g^{j_1} g^{j_2} \ldots\ldots g^{j_{2\alpha-3} \cdot j_{2\alpha-2}} .$$

Definition 1 : If (n-1) covariant vectors $L_{\alpha)i}$, i = 1, 2, ...,n-1 are linearly independent, the Finsler space is called strongly non-Riemannian.

Assume that above n-covectors $L_{\alpha)i}$ are linearly independent and put $e^i_{1)} = L^i_{1)} / L = l^i$. Here and in the following we use raising and lowering of indices as $L^i_{1)} = g^{ij} L_{1)j}$.

Further put $N_{1)ij} = g_{ij} - e_{1)i} e_{1)j}$ and matrix $N_{1)} = N_{1)ij}$ is of rank (n-1). Second vector $e_{2)}$ is introduced by

$$e^i_{2)} = L^i_{2)} / L_2,$$

where L_2 is the length of $L^i_{2)}$ relative to y^i.

Next we put $N_{2)ij} = N_{1)ij} - e_{2)i} e_{2)j}$, $E^i_{3)} = N^i_{2)j} L^j_{3)}$ and so third vector $e_{3)}$ is defined by

$$e^i_{3)} = E^i_{3)} / E_3,$$

where E_2 is the length of $E^i_{2)}$ relative to y^i. The repetition of above process gives a vector $e_{r+1)}$, r=1, 2, ..., n-1 defined by

$$e^i_{r+1)} = E^i_{r+1)} / E_{r+1},$$

where $E^i_{r+1)} = N^i_{r)j} L^j_{r+1)}, E_{r+1}$ is the length of $E^i_{r+1)}$ relative to y^i and $N_{r)ij} = N_{r-1)ij} - e_{r)i} e_{2)j}$.

Definition 2 : The orthonormal frame {e_i}, i = 1, 2, ..., n as above defined in every co-ordinate neighborhood of a strongly non-Riemannian Finsler space is called the 'Miron Frame'.

Consider the Miron frame {e□}. If a tensor T^i_j of (1, 1)-type, for instance, is given then

$$T^i_j = T_{\alpha\beta} e^i_{\alpha)} e_{\beta)j},$$

where the scalars $T_{\alpha\beta}$ are defined as

$$T_{\alpha\beta} = T^i_j e_{a)i} e^j_{a)}.$$

These scalars $T_{\alpha\beta}$ are called the scalar components T^i_j of with respect to Miron frame.

Let $H_{\alpha)\beta\gamma}$ be scalar components of the h-covariant derivatives $e^i_{\alpha)|j}$ and $V_{\alpha)\beta\gamma} / L$ be scalar components of the v-covariant derivatives $e^i_{\alpha)||j}$ with respect to $C\Gamma$ of the vector $e^i_{a)}$ belonging to the Miron frame. Then

International Journal of Recent Development in Engineering and Technology
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$$e^i_{\alpha)|j} = H_{\alpha)\beta\gamma} e^i_{\beta)} e_{\gamma)j}, \qquad e^i_{\alpha)|j} = V_{\alpha)\beta\gamma} e^i_{\beta)} e_{\gamma)j},$$

where the scalars $H_{\alpha)\beta\gamma}$ and $V_{\alpha)\beta\gamma}$ satisfy the following relations [18] :

$$H_{1)\beta\gamma} = 0, \; H_{\alpha)\beta\gamma} = -H_{\beta)\alpha\gamma}$$

$$V_{\alpha)\beta\gamma} = \delta_{\beta\gamma} - \delta^1_\beta \delta^1_\gamma, \; V_{\alpha)\beta\gamma} = -V_{\beta)\alpha\gamma}$$

Definition 3 : The scalars $H_{\alpha)\beta\gamma}$ and $V_{\alpha)\beta\gamma}$ are called connections scalars.

If $C_{\alpha\beta\gamma} / L$ be the scalar components of the (h)hv-torsion tensor C^i_{jk} i.e.,

$$LC^i_{jk} = C_{\alpha\beta\gamma} e^i_{\alpha)} e_{\beta)j} e_{\gamma)k},$$

then we have [13] :

Proposition 12.1 : $C_{1\beta\gamma} = 0, \; C_{2\mu\mu} = LC, \; C_{3\mu\mu} = \dots \dots \dots = C_{\eta\mu\mu} = 0$ for $n \geq 3$, where C is the length of C^i.

Now, we consider scalar components of covariant derivatives of a tensor field, for instance, T^i_j. Let $T_{\alpha\beta;\gamma}$ and $T_{\alpha\beta;\gamma} / L$ be the scalar components of h-and v-covariant derivatives with respect to $C\Gamma$ respectively of a tensor T^i_j i.e.,

(12.1) $\quad T^i_{j|k} = T_{\alpha\beta,\gamma} e^i_{\alpha)} e_{\beta)j} e_{\gamma)k}$ and

(12.2) $\quad LT^i_j\big|_k = T_{\alpha\beta,\gamma} e^i_{\alpha)} e_{\beta)j} e_{\gamma)k}$, then we have [34].

(12.3) $\quad T_{\alpha\beta,\gamma} = (\delta_k T_{\alpha\beta}) e^k_{\gamma)} + T_{\mu\beta} H_{\mu)\alpha\gamma} + T_{\alpha\mu} H_{\mu)\beta\gamma}$ and

(12.4) $\quad T_{\alpha\beta;\gamma} = L(\dot{\partial}_k T_{\alpha\beta}) e^k_{\gamma)} + T_{\mu\beta} V_{\mu)\alpha\gamma} + T_{\alpha\mu} V_{\mu)\beta\gamma}$.

The scalar components $T_{\alpha\beta,\gamma}$ and $T_{\alpha\beta;\gamma}$ are called h-and v-scalar derivative of $T_{\alpha\beta}$ respectively.

Two dimensional Finsler space :

The Miron frame $\{e_{1)}, e_{2)}\}$ is called the Berwald frame. The first vector $e^i_{1)}$ is the normalized supporting element $l^i = y^i / L$ and the second vector $e^i_{2)} = m^i$ is the unit vector orthogonal to l^i. If C^i has non-zero length *C*, the $m^i = \pm C^i / C$. The connection scalars $H_{\alpha)\beta\gamma}$ and $V_{\alpha)\beta\gamma}$ of a two-dimensional Finsler space are such that [18]

$H_{\alpha)\beta\gamma} = 0,\ V_{\alpha)\beta 1} = 0,\ H_{\alpha)\beta 2} = \delta^{12}_{\alpha\beta}$, which implies

(12.5) $$l^i_{|j} = 0,\ m^i_{|j} = 0,\ Ll^i\big|_j = m^i m_j,\ Lm^i\big|_j = -l^i m_j.$$

There is only one surviving scalar components of LC_{ijk} namely C_{222}. If we put $I = C_{222}$. Then $LC_{ijk} = Im_i m_j m_k$.

The scalar *I* is called the main scalar of a two-dimensional Finsler space.

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