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Analytical Study of Relativistic Space & its Consequences

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***Abstract*-- This paper presents the Analytical study of Relativistic Space and its consequences. Here, we consider Isotropic Space – Time. We deal with the practices of a Friedmann Universe I. we highlight the energy, pressure and the History of the Universe. Here, we are concerned with Geometry and Destiny as well.**

We observe the different conditions for hyperbolic, Euclidean and Spherical Universe. In fact, if Universe is open, it corresponds to the hyperbolism. If Universe is flat, it could be Euclidean case. Similarly, we come to a conclusion of spherical case, when Universe is closed.

***Keywords*--Friedmann Universe I, Flat Universe and Closed Universe, Geometry and Destiny, History of Universe, Isotropic Space – Time, and Open Universe.**

I. INTRODUCTION

The present work is an attempt towards the generalization of work done Akhundev, M.D. (1), Bohm, D; W.A. Benjamin (2) and Maurya, S.K and Gupta, K. (5). In fact, the present area is the extension of work done Eddington, A.S.(3), FOCK, V.A.(4), Synge, J.L. (6,7) and Zeldo Vich et.al.(8). In this paper, we have studied Analytically relativistic space and its consequences.

II. MATHEMATICAL TREATMENT OF THE PROBLEM

[A.] We focus on one set of solutions to the Einstein Field Equations which apply in a very particular regime. We solve them for the whole Universe under the assumption that it is an isotropic space - time.

In fact, we take that statement to an extreme and assume that at any given time, the Universe looks exactly the same at every single point in space.

There is another assumption that takes into account the extreme regularity of the Universe and that is the fact that, at any given point in space, the Universe looks very much the same in whatever direction we look. Again such an assumption can be taken to an extreme so that at any point, the Universe look exactly the same, whatever direction one look. Such a space time is dubbed to be isotropic.

Isotropy restrict the metric to work in the Einstein field equation. Initially, they must be independent of space, and solely functions of time.

Furthermore, we restrict spaces of constant curvature of which there are only three: a flat euclidean space, a positively curved space and a negatively curved space.

The metric for a flat Universe takes the following form:

$$ds^2 = -c^2\, dt^2 + a^2(t)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2]$$

We consider a(t) the scale factor and t is normally called cosmic time or physical time and

$$g_{00} = -1 \qquad g_{ij} = a^2(t)\delta_{ij}$$
$$T_{00} = \rho c^2 \qquad T_{ij} = a^2 P\, \delta_{ij}$$

We proceed with this metric and energy-momentum tensor, the Einstein field equations aregreatly simplified. We first calculate the connection coefficients. We have that the only non-vanishing elements are $\frac{d}{dt}$:

$$\tau^0_{ij} = \frac{1}{c} a\dot{a}\delta_{ij}$$

$$\tau^i_{0j} = \frac{1}{c}\frac{\dot{a}}{a}\delta^i_j$$

and the resulting Ricci tensor is

$$R_{00} = -\frac{3}{c^2}\frac{\ddot{a}}{a}$$

$$R_{0i} = 0$$

$$R_{ij} = \frac{1}{c^2}\,(a\ddot{a} + 2\dot{a}^2)\,\delta_{ij}$$

Again, the Ricci tensor is diagonal. We calculate the Ricci scalar:

$$R = -R_{00} + \frac{1}{a^2}R_{ii} = \frac{1}{c^2}\left[6\frac{\ddot{a}}{a} + 6\left(\frac{\dot{a}}{a}\right)^2\right]$$

To find the two Einstein Field equations:

$$G_{00} = R_{00} - \frac{1}{2}Rg_{00} = \frac{8\pi G}{c^4}T_{00} \iff 3\left(\frac{\dot{a}}{a}\right)^2 = 8\pi G\rho$$

$$G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij} = \frac{8\pi G}{c^4}T_{ij} \iff$$

$$-2a\ddot{a} - \dot{a}^2 = \frac{8\pi G}{c^2}a^2 P$$

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We use the first equation to simplify the 2nd equation to

$$3\frac{\ddot{a}}{a} = -4\pi G\left(\rho + 3\frac{P}{c^2}\right)$$

These two equations find the scale factor, $a(t)$, evolves as a function of time. The first equation is known as the *Friedmann-Robertson-Walker* equation or FRW equation and the metric is one of the three FRW metrics. The latter equation in $\ddot{a}$ is known as the *Raychauduri* equation.

Both of the evolution equations obtained are sourced by ρ and P. These quantities satisfy a conservation equation that arises from

$$\nabla_\alpha T^{\alpha\beta} = 0$$

And in the homogeneous and isotropic case becomes

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) = 0$$

$$and,\ P = w\rho c^2 \qquad (1)$$

where w is a constant, the *equation of state* of the system.

[B]We explore the properties of this evolving Universe. We pick two objects (galaxies for example) that lie at a given distance from each other. At time t_1 they are at a distance r_1 while at a time t_2, they are at a distance r_2. We have time interval, the change between r_1 and r_2 is given by

$$\frac{r_2}{r_1} = \frac{a(t_2)}{a(t_1)}$$

And, because of the cosmological principle, this is true for two points which chosen. It makes to parameterize the distance between the two points as

$$r(t) = a(t)\,x$$

Where x is completely independent of t. Thus we have the metric for a homogeneous and isotropic space time. It is the set of coordinates (x^1, x^2, x^3) that remain unchanged during the evolution of the Universe. We have the real, *physical* coordinates are multiplied by $a(t)$ but (x^1, x^2, x^3) are time independent and are known as *conformal* coordinates. We have the two objects considered are moving away from each other. We have relative velocity given by

$$v = \dot{r} = \dot{a}x = \frac{\dot{a}}{a}ax = \frac{\dot{a}}{a}r \equiv Hr$$

Thus, a set of spectra measured in laboratory (top panel) on a distant galaxy (bottompanel).

We measure velocities in an expanding universe. We consider a photon with wave length λ being emitted at one point and observed at some other point. We have the Doppler shift given by

$$\lambda' \simeq \lambda\left(1 + \frac{v}{c}\right)$$

Which is in a differential form

$$\frac{d\lambda}{\lambda} \simeq \frac{dv}{c} = \frac{\dot{a}}{a}\frac{dr}{c} = \frac{\dot{a}}{a}dt = \frac{da}{a}$$

And integrate form $\lambda \propto a$. We therefore have wave lengths are stretched with the expansion of the Universe. We define the factor by which the wavelength is stretched by

$$z = \frac{\lambda_r - \lambda_c}{\lambda_c} \longrightarrow 1+ \equiv \frac{a_0}{a}$$

Where a_0 is the scale factor where $a_0 = 1$. We call z the *redshift*.

For example, we see the spectra measured from a galaxy; a few lines are clearly visible and identifiable. Measured in the laboratory on Earth (top panel), we have a specific set of wavelengths but measured in a specific, distant, galaxy (bottom panel) which will be shifted to longer wavelengths. Hence a measurement of the red shift (or blue shift), i.e. a measurement of the Doppler shift, will be a direct measurement of the velocity of the galaxy.

[C] Now we have restricted ourselves to a flat Universe with Euclidean geometry. We move away from such spaces we revisit the metric. We have

$$ds^2 = -c^2dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$$

We now transform to spherical polar coordinates

$$x = r\cos\varphi\sin\theta$$
$$y = r\sin\varphi\sin\theta$$
$$z = r\cos\theta$$

and re write the metric

$$ds^2 = -c^2dt^2 + a^2(t)(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2)$$

We have FRW and Raychauduri equations in this coordinate system. We now consider a 3 dimensional surface that is positively curved.

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In other words, it is the surface of a 3 dimensional hypersphere in a fictitious space with 4 dimensions. The equation for the surface of a sphere in this 4 dimensional space, with coordinates(X, Y, Z, W), is

$$X^2 + Y^2 + Z^2 + W^2 = R^2$$

Now is the same to construct spherical coordinates in three dimensions, we build hyperspherical coordinates in 4 dimensions:

$$X = R\sin\chi\sin\theta\cos\varphi$$

$$Y = R\sin\chi\sin\theta\sin\varphi$$

$$Z = R\sin\chi\cos\theta$$

$$W = R\cos\chi$$

We work out the line element on the surface of this hyper-sphere as

$$ds^2 = dX^2 + dY^2 + dZ^2 + dW^2 = R^2[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\varphi^2)$$

This is different from the flat geometry. We transform $R\sin\chi$ into r for it we have

$$dX^2 = \frac{dr^2}{R^2 - r^2}$$

We repeat this for 3-D surface with negative curvature- a hyper-hyperboloide. In our fictitious 4-D space (not with space time), we have the surface defined by

$$X^2 + Y^2 + Z^2 - W^2 = -R^2$$

Which change good coordinate system for that surface:

$$X = R\sinh\chi\sin\theta\cos\varphi$$

$$Y = R\sinh\chi\sin\theta\sin\varphi$$

$$Z = R\sinh\chi\cos\theta$$

$$W = R\cosh\chi$$

The line element on that surface will be

$$ds^2 = dX^2 + dY^2 + dZ^2 + dW^2 = R^2[d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\varphi^2)$$

We replace $R\sinh\chi$ by r to get

$$dX^2 = \frac{dr^2}{R^2 + r^2}$$

We clearly write all three space time metrics (flat, hyperspherical, hyper-hyperbolic) in a unified way. We take $r = R\sin\chi$ for the positively curved space and $r = R\sinh\chi$ for the negatively curved space we have

$$ds^2 = -c^2dt^2 + a^2(t)\left[\frac{d^2r}{1-kr^2} + r^2(d\theta^2 + sin^2\theta d\varphi^2)\right] \text{----} (2)$$

where k is positive, zero or negative for spherical, flat or hyperbolic geometries, and $k = 1/R^2$. We repeat the calculation undertook for a flat geometry and find the connection coefficients, Ricci tensor and scalar and the evolution equations. We take the metric $g_{\alpha\beta} = diag\left(-1, \frac{a^2}{1-kr^2}, a^2r^2, sin\theta^2\right)$ and we have for this of coordinates, the i and j labels run over r, θ and φ. Thus, we find the connection coefficients are :

$$\tau^0_{ij} = \frac{1}{c}a\dot{a}\,\tilde{g}_{ij}$$

$$\tau^i_{0j} = \frac{1}{c}\frac{\dot{a}}{a}\delta^i_j$$

$$\tau^i_{jk} = \tilde{\tau}^i{}_{jk}$$

Where $\tilde{g}_{ij}$ and $\tilde{\tau}$ are the metric and connection coefficients of the conformal 3-space (that is of the 3-space with the conformal factor, a, divided out) :

$$\tilde{\tau}^r{}_{rr} = \frac{kr}{1-kr^2}$$

$$\tilde{\tau}^r{}_{\theta\theta} = -r(1-kr^2)$$

$$\tilde{\tau}^r{}_{\varphi\varphi} = -(1-kr^2)r\sin^2(\theta)$$

$$\tilde{\tau}^\theta{}_{\theta r} = \frac{1}{r}$$

$$\tilde{\tau}^\theta{}_{\varphi\varphi} = \frac{-\sin(2\theta)}{2}$$

$$\tilde{\tau}^\varphi{}_{\varphi r} = \frac{1}{r}$$

$$\tilde{\tau}^\varphi{}_{\theta\varphi} = \frac{1}{\tan(\theta)}$$

The Ricci tensor and scalar can be combined to form the Einstein tensor

$$G_{00} = 3\,\frac{\dot{a}^2 + kc^2}{c^2a^2}$$

$$G_{ij} = -\frac{2a\ddot{a} + \dot{a}^2 + kc^2}{c^2}\tilde{g}_{ij} \text{ ------- (3)}$$

while the energy-momentum tensor is

$$T_{00} = \rho c^2$$

$$T_{ij} = a^2P\tilde{g}_{ij}$$

Combining them give us the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$$

while the Raychauduri equations remains as

$$3\frac{\ddot{a}}{a} = \frac{4\pi G}{3}\left(\rho + 3\frac{P}{c^2}\right)$$

We explore the consequences of the overall geometry of the Universe, i.e. the term proportional to k in the FRW equations: For simplicity, we consider a dust filled universe. We see that the term proportional to k will only be important at late times, when it dominates over the energy density of dust. In other words, in the universe we say that *curvature dominates* at late times. We consider the two possibilities. First of all, let us take $k < 0$. We then have

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{\lfloor k \rfloor c^2}{a^2}$$

When the curvature dominates we have

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\lfloor k \rfloor c^2}{a^2}$$

So $a \propto t$. In this case, the scale factor grows at the speed of light. We also consider $k > 0$. From the FRW equations we see, there is a point, when $\frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$ and therefore $\dot{a} = 0$. When the Universe stops expanding, at this point the Universe starts contracting and evolves to a *Big Crunch.* Clearly geometry is intimately tied to destiny,

If k = 0, there is a strict relationship between $H = \frac{\dot{a}}{a}$ and ρ. Indeed from the FRW equation we have

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \rightarrow \rho = \rho_c \equiv \frac{3H^2}{8\pi G}$$

We call ρ_c the critical density. It is a function of a. If we take $H_0 = 100hKm\, s^{-1}\, Mpc^{-1}$, we have

$$\rho_c = 1.9 \times 10^{-26}\, h^2 kgm^{-3}$$

Which corresponds to a few atoms of Hydrogen per cubic meter. We compare this with the density of water is $10^3\, kgm^{-3}$. Now we take another look at the FRW equation and re-write it as

$$\frac{1}{2}\dot{a}^2 - \frac{4\pi G}{3}\rho a^2 = -\frac{1}{2}\, kc^2$$

Which has the form $E_{tot} = U + K$ and we equate E_{tot} to - kc^2, so that K is the kinetic energy, is the gravitational energy. We see that if $\rho = \rho_c$, it corresponds to the total energy of the system being 0, i.e., kinetic and gravitational energy balance themselves out perfectly.

In the case of nonzero k, $k < 0 \;\; \rho < \rho_c$ and therefore total energy is positive, kinetic energy wins out and the Universeexpands at a constant speed.

$k > 0 \;\; \rho > \rho_c$ and the total energy is negative, gravitational energy wins out and the Universe recollapses.

We now have the geometry related to the energy density.

For the *fractional energy density* or *densityparameter*. We define

$$\Omega \equiv \frac{\rho}{\rho_c}$$

It will be a function of a and we normally express its value at presence as Ω_0. If there are various contributions to the energy density, we define the fractional energy densities of each one of these contributions. For example

$$\Omega_R \equiv \frac{\rho_R}{\rho_c} \quad \Omega_M \equiv \frac{\rho_M}{\rho_c} \quad .$$

We define two additional Ω_s :

$$\Omega_\Lambda \equiv \frac{\Lambda}{3H^2}$$

$$\Omega_k \equiv -\frac{kc^2}{a^2H^2}$$

and we have Ω :

$$\Omega = \Omega_R + \Omega_M + \Omega_\Lambda$$

We now have –

$\mathbf{\Omega < 1 \;:\; \rho < \rho_c\,, k < 0}$, *Universe is open* (**hyperbolic**)

$\mathbf{\Omega = 1 \;:\; \rho = \rho_c\,, k = 0}$, **Universe is flat (Euclidean)**

$\mathbf{\Omega > 1 \;:\; \rho > \rho_c\,, k > 0}$, *Universe is closed* (**spherical**)

Hence, the result

III. Conclusions

We have investigated three results as Universe is Open (Hyperbolic), Universe is flat (Euclidian) and Universe is closed (Spherical) at certain conditions.

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