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Analytical Study on the Dual of Locally Convex Algebra

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***Abstract*--This present paper deals with the study on the Dual of Locally Convex Algebras. Here, we consider a property of equicontinuous subsets of the dual of $E \underset{\in}{\otimes} F$, where E and F are the Locally Convex algebras which are the inductive limits of the families (E_i) and (F_j) respectively of Locally Convex Algebras by the linear mappings (f_i) and (h_j) respectively, it is proved in this paper that $E \underset{\in}{\otimes} F$ denotes the $\in$ - tensor product of E and F.**

***Keywords* – Dual Space, Locally Convex Algebras, Equicontinuous Space, Inductive Tensor Product, Linear Mapping.**

I. INTRODUCTION

The present work is an attempt towards the generalization of work done by E.G. EFFROS (1) and Kothe (3,4). In fact, the present area is the extension of work done by Halub, J.R.(2), Kumar et al. (5), Srivastava et al (6), Srivastava et al (7), Srivastava et al (8), Srivastava et al. (9) and Srivastava et al. (10). In this paper we have studied analytically on dual of locally convex algebras.

II. MATHEMATICAL TREATMENT OF THE PROBLEM

<u>Proposition</u> : Let E be a locally convex algebra which is the inductive limit of a family (E_i) of locally convex (not necessarily Hausdorff) algebras by the family (f_i) of linear mappings. Let F be a locally convex algebra which is the inductive limit of a family (F_j) of locally convex (not necessarily Hausdorff) algebras by the family (h_j) of linear mappings. Let T denote the finest locally convex topology on $E \otimes F$ for which the linear maps

$$F_i \otimes h_j : E_i \underset{\in}{\otimes} F_j \rightarrow E \otimes F$$

are continuous and let $E \otimes F$ with such a topology T be denoted by $E \underset{T}{\otimes} F$.

Then every equicontinuous subset M of $(E \underset{\in}{\otimes} F)'$ is also an equicontinuous subset of $(E \underset{T}{\otimes} F)'$.

Proof : It is well known that an inductive limit of locally convex Hausdorff algebras need not be Hausdorff. We therefore divide the proof into parts.

Case I : If all the locally convex algebras E_i, F_j are Hausdorff algebras and their inductive limits E and F are also Hausdorff algebras then the proof is trivial on the ground that the maps

$$F_i \otimes h_j : E_i \underset{\in}{\otimes} F_j \rightarrow E \underset{\in}{\otimes} F$$

are continuous. Hence the topology T is finer than $\in$ and consequently $\in$- equicontinuity of M implies its T-equicontinuity.

Case-II : Let the algebras E_i and F_j be not necessarily Hausdorff. A neighbourhood base in E is given by the sets of the form

$$L \bigcup_{i} f_i(V_i),$$

where for each i, V_i is a basic neighbourhood in E_i. Let us note that indicates the absolutely convex envelope.

Similarly a neighbourhood base in F is given by the sets of the form

$$L \bigcup_{j} h_j(W_j),$$

where for each j, W_j is a basic neighbourhood in F_j.

A base of neighbourhood in $E \underset{T}{\otimes} F$ is formed by the sets of the form

$$L \bigcup_{i,j} f_j \otimes h_j (V_i^0 \otimes W_j^0)^0 ,$$

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where the polars V_i^0 are taken in E_i' and the polars W_j^0 are taken in F_j'. Now recalling that

$$E_i' \otimes F_j' \subseteq (E_i \otimes_{\in} F_j)' ,$$

the polars $(V_i^0 \otimes W_j^0)^0$ are taken in $E_i \otimes_{\in} F_j$.

Let us note that $f_i \otimes h_j$ denotes the tensor product of the linear mappings f_i and h_j. A base of neighbourhoods in $E \otimes_{\in} F$ is formed by the sets of the form

$$\left(\left(L \bigcup_i f_i(V_i) \right)^0 \otimes \left(L \bigcup_j h_j(W_j) \right)^0 \right)^0$$

Now let M be an equicontinuous subset of $(E \otimes_{\in} F)'$. Then

$$M \subseteq \left(\left(L \bigcup_i f_i(V_i) \right)^0 \otimes \left(L \bigcup_j h_j(W_j) \right)^0 \right)^{00} \ldots\ldots\ldots \qquad (1)$$

for some choice of V_i's and W_j's.

We shall prove that for such a choice of V_i and W_j,

$$M \subseteq \left(\left(L \bigcup_{i,j} f_i \otimes h_j (V_i^0 \otimes W_j^0) \right)^0 \right)^0 \ldots\ldots\ldots\ldots \qquad (2)$$

which will imply that M is an equicontinuous subset of $(E \otimes_T F)'$.

It is sufficient to prove that

$$\left(\left(L \bigcup_i f_i(V_i) \right)^0 \otimes \left(L \bigcup_j h_j(W_j) \right)^0 \right)^{00}$$

$$\subseteq \left(L \bigcup_{i,j} f_i \otimes h_j (V_i^0 \otimes W_j^0)^0 \right)^0 \ldots\ldots\ldots\ldots \qquad (3)$$

Now applying the "theorem of bipolars" it is sufficient to prove

$$\overline{L \left(\left(L \bigcup_i f_i(V_i) \right)^0 \otimes \left(L \bigcup_j h_j(W_j) \right)^0 \right)}$$

$$\subseteq \left(L \bigcup_{i,j} f_i \otimes h_j (V_i^0 \otimes W_j^0)^0 \right)^0 \ldots\ldots\ldots\ldots\ldots \qquad (4)$$

where the line segment above L is used to denote the "closure".

Now since the right hand set in "$\subseteq$" in (4) is weakly

closed and absolutely convex, it is sufficient to prove that

$$\left(L \bigcup_i f_i(V_i) \right)^0 \otimes \left(L \bigcup_j h_j(W_j) \right)^0$$

where the polar $\left(\left(L \bigcup_i f_i(V_i) \right)^0 \right.$ is taken in E' and $\left(L \bigcup_j h_j(W_j) \right)^0$ is taken in F', and recalling that

$$E' \otimes F' \subseteq (E \otimes_{\in} F)'$$

the final polar is taken in $E \otimes_{\in} F$.

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$$\subseteq \left(L \bigcup_{i,j} f_i \otimes h_j(V_i^0 \otimes W_j^0)^0 \right) \qquad \ldots\ldots \quad (5)$$

Now if f_i' and h_j' denote respectively the transposes of f_i and h_j then we have

$$(L \bigcup_i f_i(V_i))^0 \subseteq f_i'^{-1}(V_i^0),$$

$$(L \bigcup_j h_j(W_j))^0 \subseteq h_j'^{-1}(W_j^0),$$

therefore we have

$$(L \bigcup_i f_i(V_i))^0 \otimes (L \bigcup_j h_j(W_j))^0$$

$$\subseteq f_i'^{-1}(V_i^0) \otimes h_j'^{-1}(W_j^0).$$

Hence for proving (5), it is sufficient to prove that

$$f_i'^{-1}(V_i^0) \otimes h_j'^{-1}(W_j^0)$$

$$\subseteq \left(L \bigcup_{i,j} f_i \otimes h_j(V_i^0 \otimes W_j^0)^0 \right) \qquad \ldots\ldots \quad (6)$$

Now let $z = f_i'^{-1}(a_i) \otimes h_j'^{-1}(b_j)$ be any element of the left hand set of "$\subseteq$" in (6) and $a_i \in V_i^0$, $b_j \in W_j^0$.

Any element of $L \bigcup_{i,j} f_i \otimes h_j(V_i^0 \otimes W_j^0)^0$ is of the form

$$x = \sum_{ij} a_{ij} f_i \otimes h_j(x_{ij}),$$

where $\sum_{ij} |a_{ij}| \leq 1$, $a_{ij} = 0$ except for finitely many i and j and

$x_{ij} \in (V_i^0 \otimes W_j^0)^0$. Then

$$|\langle z,x\rangle| = |\langle f_i'^{-1}(a_i) \otimes h_j'^{-1}(b_j), \sum a_{ij} f_i \otimes h_j(x_{ij})\rangle|$$

$$= |\sum a_{ij} \langle f_i'^{-1}(a_i) \otimes h_j'^{-1}(b_j), f_i \otimes h_j(x_{ij})\rangle|$$

$$= |\sum a_{ij} \langle a_i \otimes b_j, (f_i'^{-1} \otimes h_j'^{-1})'(f_i \otimes h_j)(x_{ij})\rangle|$$

$$= |\sum a_{ij} \langle a_i \otimes b_j, x_{ij}\rangle|$$

$$\leq \sum |a_{ij}| \,.\, |\langle a_i \otimes b_j, x_{ij}\rangle|$$

$$\leq \sum |a_{ij}| \leq 1$$

Hence $z \in \left(L \bigcup_{i,j} f_i \otimes h_j(V_i^0 \otimes W_j^0)^0 \right)^0$.

Thus we have established the inclusion "$\subseteq$" in (6).
This completes the proof. Hence the result.

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REFERENCES

[1] E.G. Effros and Algebras. E.C. Lance . (1970) Tensor Products of Operator

[2] Halub, J.R. (1970) Tensor Product Mapping Math Ann ., Vol. 188, pp 01-12 .

[3] Kothe, G. (1969) Topological Vector Spaces , Springer – Verlag , I,New York.

[4] Kothe, G. (1979) Topological Vector Spaces , Springer – Verlag , II, New York.

[5] Kumar Nirmal; Choudhary, D., Talukdar, A.U., Srivastava, U.K., (2016) International Journal of Emerging Technology and Advanced , Engineering Vol. 6(11), pp 170-173.

[6] Srivastava , U.K., Kumar, S., and Singh, T.N. (2011) J.P.A.S (Mathematical Science) ,Vol.17, pp 188-191.

[7] Srivastava, U.K., Kumar, N., Kumar ,S., and Singh, T.N. (2012) J.P.A.S (Mathematical Science) ,Vol. 18,pp 189- 191.

[8] Srivastava , U.K., kumar,N., Kumar , S., and Singh ,T.N. (2012) Proc. Of Math. Soc., B.H.U. Vol. 28,pp 25-28.

[9] Srivastava, U.K., Talukdar,A.U., Shahabuddin, Md., and Pandit, A.S., (2016) International Journal of Emerging Technology and Advanced Engineering Vol. 6(6), pp 247-250.

[10] Srivastava, U.K., Roy, L.K.;Prasad, B; Pathak, C.D., and Ray, S. (2020) Journal of Mathematics and statistical science, U.S.A. Vol. 06 Issue (05), pp. 182 – 186.