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Comparison of Magnetostatic Field Calculations Associated with Thick Solenoids in the Presence of Iron Using the Legendre's Complete Elliptic Integrals of the 1st And 2nd Kind and the Euler-Maclaurin Summation Formula.

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***Abstract*— The effect of iron on the uniformity of the field produced by an axisymmetric thick solenoid is considered. Here two solution to the vector potential and hence the magnetic field components will be derived. The first solution is obtained using the complete elliptic integrals of Legendre; the other is obtained using the Euler-Maclaurin summation formula, thus converting the doubly infinite summation into an integral. Numerical results are presented as are the field distribution.**

***Index Terms*— Time independent field, Elliptic Integrals of the first and second kind, the Euler-Maclaurin Summation formula. Component.**

I. Introduction

In this paper magnetostatic field calculations associated with an axisymmetric conductor of rectangular cross section situated equidistant from two semi-infinite regions of iron of finite permeability are computed. The magnetostatic field associated with iron-free axisymmetric systems has been considered by Boom and Livingstone [2], Garrett [3] and many others. Caldwell [4], Caldwell and Zisserman [5] and [6] have carried out work which takes account of the effects of the presence of iron on such systems. The main advantages of introducing iron are:

i. Higher fields are provided for the same current, producing substantial power savings over conventional conductors.
ii. The field uniformity is improved even for superconducting solenoids by placing the iron in a suitable position.

The geometry considered is shown in figure 1, a toroidal conductor V' of rectangular cross section having inner radius A, outer radius B and length L-2ε, is located equidistant between two semi-infinite regions of iron of finite permeability a distance L apart, the axis of the torus being perpendicular to the iron boundaries. The region V between the conductor and the iron is assumed insulating. Cylindrical polar coordinates *(r,ϕ, z)* are used where r and z are normalized in terms of L.

Prior to Caldwell [3] the presence of iron in axisymmetric systems had been largely ignored see Loney [8] and Garrett [3] et al. In cylindrical polar coordinates Maxwell's equations give:

$$\underline{\nabla} \wedge \underline{B} = \begin{cases} 0 & in\ V \\ -Ce_{\phi} & in\ V' \end{cases}$$

where e_{ϕ} is a unit vector in the direction of increasing ϕ and C is a constant with

$$\underline{\nabla}.\underline{B} = 0 \text{ in V and V'}. \qquad (1)$$

Equation (1) suggests the introduction of a potential $\underline{A}$ such that $\underline{B} = \underline{\nabla} \wedge \underline{A}$, axial symmetry implies

$$B_r = -\frac{\partial A_{\phi}}{\partial z};\ B_{\phi} = 0; B_z = \frac{1}{r}\frac{\partial (rA_{\phi})}{\partial r}$$

By Maxwell's equation:

$$\underline{\nabla} \wedge \underline{B} = \underline{\nabla} \wedge (\underline{\nabla} \wedge \underline{A}) = \begin{cases} 0 & in\ V \\ Ce_{\phi} & in\ V' \end{cases}$$

thus

$$\frac{1}{r}\begin{vmatrix} e_r & e_{\phi} & e_z \\ \frac{\partial}{\partial r} & 0 & \frac{\partial}{\partial z} \\ -\frac{\partial A_{\phi}}{\partial z} & 0 & \frac{1}{r}\frac{\partial (rA\phi)}{\partial r} \end{vmatrix} = \begin{cases} 0 & in\ V \\ -Ce_{\phi} & in\ V' \end{cases}$$

$$\Rightarrow \nabla_1^2 A_{\phi} = \begin{cases} 0 & in V \\ Ce_{\phi} & in V' \end{cases}$$

where $\nabla_1^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2}$

With boundary conditions

$A_\phi = 0$ on r = 0

$A_\phi \to 0$ as $r \to \infty$

$\frac{\partial A_\phi}{\partial z} = 0$ on $z=0$ and $z=1$

Using the integral representation of the vector potential this gives:

$\underline{A}(\underline{r}) = \int_{V'} \frac{j(\underline{r})}{|\underline{r} - \underline{r}'|} dv'$, hence for finite μ,

$$A_\varphi(r,z) = \frac{\mu_0 j}{4\pi} \sum_{n=-\infty}^{\infty} K^{|n|} \int_a^b \int_0^{2\pi} \int_\varepsilon^{1-\varepsilon} \frac{x\cos\vartheta}{|\underline{r} - \underline{r}'|} dx d\vartheta dz'$$

(2)

in cylindrical coordinates

$$|\underline{r} - \underline{r}'| = ((z-z')^2 + r^2 + x^2 - 2xr\cos\vartheta)^{1/2}$$

With $K = \frac{\mu - 1}{\mu + 1}$, known as the image factor.

II. The Solution To The Magnetic Vector Potential Using The Complete Elliptic Integrals

In order to obtain the solution to the vector potential $A_\varphi(r,z)$ and hence the field components $B_r(r,z)$ and $B_z(r,z)$ use of the complete elliptic integrals of Legendre is made. Using expression (2) if integration with respect to ϑ is done first the complete Elliptic Integrals of Legendre of the first and second kind respectively are obtained. Defining

$$I_\vartheta = \int_0^{2\pi} \frac{x\cos\vartheta}{(\beta^2 - \gamma\cos\vartheta)^{1/2}} d\vartheta$$

where $\beta^2 = w^2+x^2+r^2$ and $\gamma=2xr$ so that:

$$I_\vartheta = \frac{x}{\beta}\int_0^{2\pi} \frac{\cos\vartheta}{(1-k\cos\vartheta)^{1/2}} d\vartheta$$

where k = $\frac{\gamma}{\beta^2}$, so that

$$I_\vartheta = -\frac{\beta}{2r}\int_0^{2\pi} \frac{1-k\cos\vartheta - 1}{(1-k\cos\vartheta)^{1/2}} d\vartheta$$

$$\Rightarrow I_\vartheta = -\frac{\beta}{2r}\int_0^{2\pi} (1-k\cos\vartheta)^{1/2} d\vartheta$$

$$+ \frac{\beta}{2r}\int_0^{2\pi} \frac{d\vartheta}{(1-k\cos\vartheta)^{1/2}}$$

So that with slight manipulation this can be written as

$$I_\vartheta = -\frac{2\beta^2}{r(\beta^2+\gamma)^{1/2}}((1+k)E(\delta) - K(\delta))$$

Where

$$\delta^2 = \frac{2k}{1+k}$$

Where K(δ) and E(δ) are the complete Elliptic integrals of the first and second kind respectively. Provided 0<δ<1 these integrals may be expressed as a series which is uniformly convergent and thus may be integrated term by term. So considering this inequality with:

$k = \frac{2xr}{w^2+x^2+r^2}$ and $\delta^2 = \frac{2k}{1+k} = \frac{4xr}{w^2+(x+r)^2}$

hence for convergence $0 < \left(\frac{4xr}{w^2+(x+r)^2}\right)^{1/2} < 1$

i.e. 4xr > 0 which is true $\forall x, r > 0$. Similarly the second inequality gives $4xr < w^2 + (x+r)^2$ or $-(x-r)^2 < w^2$, which is again true $\forall x, r, w \neq 0$. So that the series is uniformly convergent. Hence using

$$K(\delta) = \frac{\pi}{2}\left(1 + \left(\frac{1}{2}\right)^2 \delta + \left(\frac{1.3}{2.4}\right)^2 \delta^2 + \left(\frac{1.3.5}{2.4.6}\right)^2 \delta^3 + O(\delta^4)\right)$$

and

$$E(\delta) = \frac{\pi}{2}\left(1 - \left(\frac{1}{2}\right)^2 \delta - \left(\frac{1.3}{2.4}\right)^2 \frac{\delta^2}{3} + \left(\frac{1.3.5}{2.4.6}\right)^2 \frac{\delta^3}{5} - O(\delta^4)\right)$$

gives

$$A_\phi(r,z) = -\frac{\mu_0 j}{4}\sum_{n=-\infty}^{\infty} K^{|n|}\int_a^b\int_{z-\varepsilon-n}^{z-1+\varepsilon-n}\frac{\beta^2}{r(\beta^2+\gamma)^{1/2}}\{k - (k+2)\left(\frac{1}{2}\right)^2\delta^2 - (k+4)\left(\frac{1.3}{2.4}\right)^2\frac{\delta^4}{3} - (k+6)\left(\frac{1.3.5}{2.4.6}\right)^2\frac{\delta^6}{5} - ....\}dxdz'$$

III. Considering The Higher Order Terms Of δ^n

Considering the order δ^0 term which will be denoted by I_0, say where

$$I_0 = \int_a^b\int_{z-\varepsilon-n}^{z-1+\varepsilon-n}\frac{\beta^2 k}{r(\beta^2+\gamma)^{1/2}}dxdz'$$

$$\Rightarrow I_0 = -2\int\int\frac{u-r}{(w^2+u^2)^{1/2}}dudw$$

Where $x + r = u$ and $w = z - z'$, so that

$$A_\varphi(r,z) = \frac{\mu_0 j}{4}\sum_{n=-\infty}^{\infty}K^{|n|}[[(u^2-2ru)*\log_e(w+(w^2+u^2)^{1/2}) + w(w^2+u^2)^{1/2} - 2rw\log_e(u+(w^2+u^2)^{1/2})]_{a+r}^{b+r}]_{z=z'-\varepsilon-n}^{z-1+\varepsilon-n} + O(\delta^2)$$

Evaluating the higher order terms as shown in Pavlika [9], it can be shown that:

$$A_\phi(r,z) = \frac{\mu_0 j}{4}\sum_{n=-\infty}^{\infty}K^{|n|}[[-rw\log_e(u+(w^2+u^2)^{1/2}) + \frac{r^2 w}{(w^2+u^2)^{1/2}} - \frac{r^3(27u^4+70w^4+102u^2w^2)}{6uw(w^2+u^2)^{3/2}} + \frac{r^4 w(59u^2+49w^2)}{3u^2(w^2+u^2)^{3/2}} - \frac{r^5(4u^6+6u^4w^2+21u^2w^4+14w^6)}{u^3w^3(w^2+u^2)^{3/2}} + \frac{10r^6 w(3u^2+2w^2)}{3u^4(w^2+u^2)^{3/2}} - \frac{5r^7(8u^8+12u^6w^2+3u^4w^4+12u^2w^6+8w^8)}{2(w^2+u^2)^{3/2}}]_{a+r}^{b+r}]_{z'=z-\varepsilon-n}^{z-1+\varepsilon-n} + O(\delta^8) \quad (3)$$

Or

$$A_\phi(r,z) = \frac{\mu_0 j}{4}\sum_{n=-\infty}^{\infty}K^{|n|}[[r\alpha_{3,1}(u,w) + r^2\alpha_{3,2}(u,w) + r^3\alpha_{3,3}(u,w) + r^4\alpha_{3,4}(u,w) + r^5\alpha_{3,5}(u,w) + r^6\alpha_{3,6}(u,w) + r^7\alpha_{3,7}(u,w)]_{a+r}^{b+r}]_{z=z'-\varepsilon-n}^{z-1+\varepsilon-n} + O(\delta^8)$$

Where the $\alpha_{i,j}$, $i=3$, $j=2,3,\ldots 7$ are defined by expression (3).

IV. Calculating The Radial And Axial Field Components.

Since $\underline{B} = \underline{\nabla}\wedge\underline{A}$, using cylindrical coordinates this gives

$$B_z(r,z) = \frac{\mu_0 j}{4}\sum_{n=-\infty}^{\infty}K^{|n|}[[2\alpha_{3,1}(u,w) + 3r\alpha_{3,2}(u,w) + 4r^2\alpha_{3,3}(u,w) + 5r^3\alpha_{3,4}(u,w) + 6r^4\alpha_{3,5}(u,w) + 7r^5\alpha_{3,6}(u,w) + 8r^6\alpha_{3,7}(u,w)]_{a+r}^{b+r}]_{z=z'-\varepsilon-n}^{z-1+\varepsilon-n} + O(\delta^8) \quad (4)$$

and differentiating with respect to z gives

$$B_r(r,z) = \frac{\mu_0 j}{4}\sum_{n=-\infty}^{\infty}K^{|n|}[[-r\{\log_e(u+(w^2+u^2)^{1/2}) + \frac{w^2}{(w^2+u^2)^{1/2}(u+(w^2+u^2)^{1/2})}\} + \frac{r^2u^2}{(w^2+u^2)^{3/2}} + \frac{r^3u(9u^4+2u^2w^2-2w^4)}{2w^2(w^2+u^2)^{5/2}} + \frac{r^4(59u^2+29w^2)}{3(w^2+u^2)^{5/2}} + \frac{3r^5u(4u^4+10u^2w^2+w^4)}{w^4(w^2+u^2)^{5/2}} + \frac{10r^6}{(w^2+u^2)^{5/2}} - \frac{25r^7w^5(15u^4+20u^2w^2+8w^4)}{2(w^2+u^2)^{5/2}}]_{a+r}^{b+r}]_{z'=z-\varepsilon-n}^{z-1+\varepsilon-n} \quad (5)$$

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Results for $A_\phi(r,z)$, $B_r(r,z)$ and $B_z(r,z)$ using expressions (3), (4) and (5) with a=0.9, b=1.1, ε = 0.05 and $\mu_0 j$ = 100 are shown in tables 1, 2 and 3 respectively.

V. Calculation Of The Field Components Using The Euler-Maclaurin Summation Formula

Here use of the Euler-Maclaurin summation will be made to convert the doubly infinite summation corresponding to the image coils to an integral. Much literature exists on the derivation of the formula thus only the final formula will be quoted. We have expression (2)

$$A_\phi(r,z)=\frac{\mu_0 j}{4\pi}\sum_{n=-\infty}^{\infty}K^{|n|}\int_a^b\int_0^{2\pi}\int_\varepsilon^{1-\varepsilon}\frac{x\cos\vartheta\, dx\, d\vartheta\, dz'}{\{(z-z'-n)^2+r^2+x^2-2xr\cos\vartheta)^{1/2}}$$

and considering the summation first i.e. defining

$$S=\sum_{n=-\infty}^{\infty}\frac{\gamma K^{|n|}}{((\alpha-n)^2+\beta^2)^{1/2}}$$

Where

$$\gamma=x\cos\vartheta,\ \beta^2=r^2+x^2-2xr\cos\vartheta\ and\ \alpha=z-z'$$

So that

$$S=\sum_{n=0}^{\infty}\frac{\gamma K^{|n|}}{((\alpha-n)^2+\beta^2)^{1/2}}+\sum_{n=0}^{\infty}\frac{\gamma K^{|n|}}{((\alpha+n)^2+\beta^2)^{1/2}}-\frac{\gamma}{(\alpha^2+\beta^2)^{1/2}}$$

Which may be written as

$$S=\sum_{n=0}^{\infty}f_1(n)+\sum_{n=0}^{\infty}f_2(n)-\frac{\gamma}{(\alpha^2+\beta^2)^{1/2}},\ say$$

$$=\sum_{n=0}^{\infty}f(n)-\frac{\gamma}{(\alpha^2+\beta^2)^{1/2}}$$

Where $f(n)=f_1(n)+f_2(n)$. So that the effect of the image coils has been separated from the main coil. To these images we apply the Euler-Maclaurin Summation formula. Considering the term

$$\sum_{n=0}^{\infty}f_1(n)=\int_0^\infty f_1(n)dn+\frac{1}{2}[f_1(0)-f_1(\infty)]$$
$$+\frac{1}{12}\left[f_1'(\infty)-f_1'(0)\right]-\frac{1}{720}[f_1'''(0)-f_1'''(\infty)]+\ldots$$

Letting

$$I_1(\alpha)=\int_0^\infty f_1(n)dn=\int_0^\infty\frac{\gamma k^n}{((\alpha-n)^2+\beta^2)^{1/2}}dn$$

$$=\int_0^\infty\frac{\gamma e^{-\delta n}}{((\alpha-n)^2+\beta^2)^{1/2}}dn$$

Where $\delta=\log_e|\frac{1}{K}|$ and $K=\frac{\mu-1}{\mu+1},\mu\neq 1$

So clearly the method will not cater for the case when $\mu=1$, but this is expected as this is the iron free situation. In order to make any progress with this integral the integrand will be expanded in a Maclaurin series in α which will be a small parameter. Thus

$$I_1(\alpha)=I_1(0)+\alpha I_1'(0)+\frac{\alpha^2}{2!}I_1''(0)+O(\alpha^3)\qquad(6)$$

So that

$$I_1(0)=\int_0^\infty\frac{\gamma e^{-\delta n}}{(n^2+\beta^2)^{1/2}}dn$$

$$=\frac{\gamma}{2}[S_0(\delta\beta)-\pi E_0(\delta\beta)-\pi N_0(\delta\beta)]$$

Where $S_\nu(z)=$ Schlafli's polynomial of order ν, $S_0(z)=0\forall z$, Watson [11].

$E_\nu(z)=$ Weber's function of order ν, Watson [11] and $N_\nu(z)=$ Neumann's function of order ν Watson [11]. So that

$$I_1(0)=\frac{\gamma}{2}[S_0(\delta\beta)-\pi E_0(\delta\beta)-\pi N_0(\delta\beta)]$$
$$+\int_0^\infty\frac{\partial}{\partial\alpha}\left(\frac{\gamma e^{-\delta n}}{((\alpha-n)^2+\beta^2)^{1/2}}\right)|_{\alpha=0}\ dn+O(\alpha^2)$$

now

$$I_0'(0)=\gamma\int_0^\infty\frac{ne^{-\delta n}}{(n^2+\beta^2)^{3/2}}dn$$

$$\Rightarrow I_1(\alpha)=\frac{\gamma}{2}[S_0(\delta\beta)-\pi E_0(\delta\beta)-\pi N_0(\delta\beta)]$$
$$+\gamma\alpha\int_0^\infty\frac{ne^{-\delta n}}{(n^2+\beta^2)^{3/2}}dn+O(\alpha^2)$$

International Journal of Recent Development in Engineering and Technology
Website: www.ijrdet.com (Volume 1, Special Issue 1, Oct 2013)

Furthermore

$$f_1(0)=\frac{\gamma}{(\alpha^2+\beta^2)^{1/2}} \text{ and } f_1(\infty)=0$$

$$f_1^{'}(0)=-\gamma.\frac{(\alpha^2\delta-\alpha+\beta^2\delta)}{(\alpha^2+\beta^2)^{3/2}} \text{ and } f_1^{'}(\infty)=0$$

So that

$$\sum_{n=0}^{\infty} f_1(n)=\frac{\gamma}{2}[S_0(\delta\beta)-\pi E_0(\delta\beta)-\pi N_0(\delta\beta)]$$
$$+\gamma\alpha\int_0^{\infty}\frac{ne^{-\delta n}}{(n^2+\beta^2)^{3/2}}dn+\frac{1}{2}.\left[\frac{\gamma}{(\alpha^2+\beta^2)^{1/2}}\right]$$
$$+\frac{\gamma}{12}\left[\frac{\alpha^2\delta-\alpha+\beta^2\delta}{(\alpha^2+\beta^2)^{3/2}}\right]+O(\alpha^2).$$

Now considering

$$\sum_{n=0}^{\infty} f_2(n)=\int_0^{\infty} f_2(n)dn+\frac{1}{2}[f_2(0)-f_2(\infty)]$$
$$\frac{1}{12}\left[f_2^{'}(\infty)-f_2^{'}(0)\right]-\frac{1}{720}[f_2^{'''}(0)-f_2^{'''}(\infty)]+\ldots$$

With similar manipulation as just described it can be shown that

$$\sum_{n=0}^{\infty} f_2(n)=\frac{\gamma}{2}[S_0(\delta\beta)-\pi E_0(\delta\beta)-\pi N_0(\delta\beta)]$$
$$-\gamma\alpha\int_0^{\infty}\frac{ne^{-\delta n}}{(n^2+\beta^2)^{3/2}}dn+\frac{1}{2}.\left[\frac{\gamma}{(\alpha^2+\beta^2)^{1/2}}\right]$$
$$+\frac{\gamma}{12}\left[\frac{\alpha^2\delta-\alpha+\beta^2\delta}{(\alpha^2+\beta^2)^{3/2}}\right]+O(\alpha^2).$$

So that

$$S=\frac{\gamma}{2}[S_0(\delta\beta)-\pi E_0(\delta\beta)-\pi N_0(\delta\beta)]$$
$$+\frac{1}{6}.\left[\frac{\gamma\delta}{(\alpha^2+\beta^2)^{1/2}}\right]+O(\alpha^2).$$

To proceed with this method these special functions must be written in a form so that they can be integrated over the volume of interest.

VI. Neumann's Function, Bessel Function Of The Second Kind

Here the Bessel function of the second kind has been obtained, taking the definition of the Neumann function as

$$N_\upsilon(x)=\frac{\cos\upsilon\pi J_\upsilon(x)-J_{-\upsilon}(x)}{\sin\upsilon\pi}$$

Where $J_n(x)$ is the usual Bessel function of the first kind of order n, evaluating $N_n(x)$ by l'Hopital's rule for indeterminant forms (i.e. for $\upsilon=n$ (integer)) gives

$$N_n(x)=\frac{1}{\pi}\left[\frac{\partial}{\partial\upsilon}J_\upsilon(x)-(-1)^n\frac{\partial}{\partial\upsilon}J_{-\upsilon}(x)\right]|_{\upsilon=n}$$

With

$$J_n(x)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(n+m+1)}.\left(\frac{x}{2}\right)^{2m+n}$$

Using

$$\frac{d}{d\nu}(x^\upsilon)=x^\upsilon\log_e(x)$$

and

$$\frac{d}{dz}(\Gamma(z)=\Gamma\frac{d}{dz}(\log_e(\Gamma(z)))$$

giving

$$N_n(x)=\frac{2}{\pi}J_n(x)\log_e(\frac{x}{2})$$
$$-\frac{1}{\pi}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!(n+r)!}.\left(\frac{x}{2}\right)^{n+2r}(F(r)+F(n+r))$$
$$-\frac{1}{\pi}\sum_{r=0}^{n-1}\frac{(n-r-1)!}{r!}.\left(\frac{x}{2}\right)^{-n+2r}$$

Where F(r) and F(n+r) are the digamma functions (Abramowitz and Stegun [1]) arising from the differentiation of the gamma function when expressed as an infinite limit. Using properties of the digamma function gives:

$$N_n(x)=\frac{2}{\pi}\left(\log_e(\frac{x}{2})+\gamma'-\frac{1}{2}\sum_{p=1}^{n}\frac{1}{p}\right)J_n(x)$$
$$-\frac{1}{\pi}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!(n+r)!}\cdot\left(\frac{x}{2}\right)^{n+2r}\sum_{p=1}^{r}\left(\frac{1}{p}+\frac{1}{p+n}\right)$$
$$-\frac{1}{\pi}\sum_{r=0}^{n-1}\frac{(n-r-1)!}{r!}\cdot\left(\frac{x}{2}\right)^{-n+2r}$$

Where γ' is the Euler-Mascheroni constant (Abramowitz and Stegun [1]). So finally for n=0 the limiting value is:

$$N_0(x)=\frac{2}{\pi}\left(\log_e(x)+\gamma'-\log_e(2)\right)+O(x^2).$$

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VII. The Weber Function And Its Relation To The Struve Function

By definition the Weber function may be expressed as

$$E_v(x)=\frac{1}{\pi}\int_0^{\pi}\sin(\upsilon\vartheta-z\sin\vartheta)d\vartheta$$

The relationship between Weber's function and the Struve function is, for n being a positive integer or zero (Abramowitz and Stegun [1])

$$E_v(x)=\frac{1}{\pi}\sum_{k=0}^{(n-1)/2}\frac{\Gamma(k+\frac{1}{2})\left(\frac{1}{2}z\right)^{n-2k-1}}{\Gamma(n+\frac{1}{2}-k)}-H_n(z)$$

Where $H_n(z)$ is the Struve function defined by

$$H_v(x)=(\frac{1}{2})^{\upsilon+1}\sum_{k=0}^{\partial\infty}\frac{(-1)^k z^{2k}}{\Gamma(k+\frac{3}{2})\Gamma(k+\upsilon+\frac{3}{2})}\left(\frac{1}{2}\right)^{2k}$$

It follows that

$$E_0(z)=-H_0(z)$$
$$\Rightarrow E_0(z)=-\frac{2}{\pi}(z-\frac{z^3}{1^2 3^2}+\frac{z^5}{1^2 3^2 5^2}-......)$$

This gives

$$S=2\gamma\delta\beta-2\gamma\left[\log_e(\delta\beta)+\gamma'-\log_e(2)\right]$$
$$+\frac{1}{6}\cdot\left[\frac{\gamma\delta}{(\alpha^2+\beta^2)^{1/2}}\right]+O(\alpha^2).$$

Where to avoid confusion the Euler-Mascheroni constant has been denoted by γ' and $\gamma=x\cos\vartheta$. Thus integration over the volume of interest can now be performed. That is

$$A_\varphi(r,z)=\frac{\mu_0 j}{4\pi}\int_a^b\int_0^{2\pi}\int_\varepsilon^{1-\varepsilon}\{2\gamma\delta\beta-2\gamma\left[\log_e(\delta\beta)+\gamma'-\log_e(2)\right]$$
$$+\frac{1}{6}\cdot\left[\frac{\gamma\delta}{(\alpha^2+\beta^2)^{1/2}}\right]\}dxd\vartheta dz'+O(\alpha^2).$$

VIII. Considering The Order $\gamma\delta$ Term In The Expression For $A_\varphi(r,z)$

Considering the $O(\gamma\delta)$ term and denoting this as

$$\Theta=\frac{\mu_0 j\delta}{24\pi}\int_a^b\int_0^{2\pi}\int_\varepsilon^{1-\varepsilon}\frac{\gamma}{(\alpha^2+\beta^2)^{1/2}}dxd\vartheta dz' \quad (7)$$

Performing the ϑ integration first gives

$$\Theta=\frac{\mu_0 j\delta}{24\pi}\int_a^b\int_\varepsilon^{1-\varepsilon}xdxdz'\int_0^{2\pi}\frac{\cos\vartheta}{(\lambda^2-\eta\cos\vartheta)^{1/2}}d\vartheta$$

Where $\lambda^2=(z-z')^2+x^2+r^2$ and $\eta=2xr$. Slight manipulation leads to

$$\Theta=\frac{\mu_0 j\delta}{24\pi}\int_a^b\int_\varepsilon^{1-\varepsilon}\frac{x}{\mu}dxdz'\int_0^{\pi/2}\frac{2\sin^2u-1}{(1-k^2\sin^2u)^{1/2}}du$$

Where

$$\mu^2=\lambda^2+\eta=(z-z')^2+(x+r)^2 \text{ and}$$

$$k^2=\frac{4xr}{(z-z')^2+(x+r)^2}, \text{ with } \frac{\vartheta}{2}=\frac{\pi}{2}-u.$$

It can be shown that (Gradsteyn and Ryzhik [7])

$$\int_0^{\pi/2} \frac{\sin^{2\mu-1} x \cos^{2\nu-1} x}{(1-k^2\sin^2 x)^\rho} dx = \frac{1}{2} B(\mu,\nu) F(\rho,\mu,\mu+\nu,k^2)$$

Where B(m,n) is the Beta function and $F(a,b,c,z^2)$ is the Hypergeometric function, so that

$$\int_0^{\pi/2} \frac{2\sin^2 u - 1}{(1-k^2\sin^2 u)^{1/2}} du = B(\frac{3}{2},\frac{1}{2}) F(\frac{1}{2},\frac{3}{2},2,k^2)$$
$$-\frac{1}{2} B(\frac{1}{2},\frac{1}{2}) F(\frac{1}{2},\frac{1}{2},1,k^2)$$

So that

$$\Theta = \frac{\mu_0 j\delta}{6\pi} B(\frac{3}{2},\frac{1}{2}) \int_a^b \int_\varepsilon^{1-\varepsilon} \frac{x}{((z-z')^2+(x+r)^2)^{1/2}} .$$
$$* F(\frac{1}{2},\frac{3}{2},2,k^2) dxdz'$$

$$-\frac{\mu_0 j\delta}{12\pi} B(\frac{1}{2},\frac{1}{2}) \int_a^b \int_\varepsilon^{1-\varepsilon} \frac{x}{((z-z')^2+(x+r)^2)^{1/2}} .$$
$$* F(\frac{1}{2},\frac{1}{2},1,k^2) dxdz', \qquad (8)$$

with

$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(n+m)}$, it can also be shown that

$B(\frac{3}{2},\frac{1}{2}) = \frac{\pi}{2}$ and $B(\frac{1}{2},\frac{1}{2}) = \pi$.

Pavlika [10] has shown that these integrals containing the series of the hypergeometric function are uniformly convergent in the interval of integration so that with some algebraic manipulation it can be shown that Pavlika [10]

$$\Theta = \frac{\mu_0 j\delta}{12} \int_a^b \int_\varepsilon^{1-\varepsilon} \{ \frac{x}{((z-z')^2+(x+r)^2)^{1/2}} .$$
$$* \sum_{n=0}^{\infty} E_n \frac{k^{2n}}{n!} \} dxdz$$

Where $E_n = C_n - D_n$ and

$C_n = \frac{(\frac{3}{2},n)(\frac{3}{2},n)}{(2,n)}, D_n = \frac{(\frac{3}{2},n)(\frac{1}{2},n)}{(1,n)}$ with

$(\lambda,k) = \frac{\Gamma(\lambda,k)}{\Gamma(\lambda)} = \lambda(\lambda+1).....(\lambda+k-1), k \geq 0$.

IX. Considering The Order k^0 Term In The Expression For Θ

Considering the term and denoting this integral as K_0 that is:

$$K_0 = \frac{\mu_0 jE_0}{12} \int_a^b \int_\varepsilon^{1-\varepsilon} \frac{x}{((z-z')^2+(x+r))^{1/2}} dxdz'$$

thus

$$K_0 = -\frac{\mu_0 jE_0}{12} [[(\frac{u^2}{2} - ru) \log_e(\sigma + (\sigma^2+u^2)$$
$$+ \frac{\sigma}{2}(\sigma^2+u^2)^{1/2}$$
$$- r\sigma \log_e(u + (\sigma^2+u^2)^{1/2}]_{a+r}^{b+r}]_{z-\varepsilon-n}^{z-1+\varepsilon-n}$$

Where u = x + r and $\sigma = z - z'$.

X. Considering The Order k^2 Term In The Expression For Θ

Considering the $O(k^2)$ term and denoting this term as K_2, say where:

$$K_2 = \frac{\mu_0 jE_1}{3} r \int_a^b \int_\varepsilon^{1-\varepsilon} \frac{x^3}{((z-z')^2+(x+r)^{3/2}} dxdz'$$

Computing these integrals gives

$$K_2 = -\frac{\mu_0 jE_1}{3} r[[w(w^2+u^2)^{1/2}$$
$$+ 3ru \log_e(w + (w^2+u^2)^{1/2})$$
$$- 3rw \log_e(w + (w^2+u^2)^{1/2}) + 3r(w^2+u^2)^{1/2}$$
$$- 3r^2 \log_e(w + (w^2+u^2)^{1/2})$$
$$+ \frac{r^3(w^2+u^2)^{1/2}}{uw})]_{a+r}^{b+r}]_{z-\varepsilon}^{z-1+\varepsilon}$$

Where u = x + r and w = z - z'. Therefore

$$A_{\varphi}(r,z) = \frac{\mu_0 j}{4\pi}\int_a^b\int_0^{2\pi}\int_{\varepsilon}^{1-\varepsilon}\{2\gamma\delta\beta - 2\gamma\left[\log_e(\delta\beta)+\gamma' - \log_e(2)\right]\}dxd\vartheta dz' + K_0 + K_2 + O(\alpha^2). \quad (9)$$

XI. Considering The Order $\delta\beta^0$ Term In The Expression For $A_{\varphi}(r,z)$.

Considering the $O(\delta\beta^0)$ term in equation (9) and denoting this term by Δ_0, say where

$$\Delta_0 = -\frac{\mu_0 j}{2\pi}(\gamma' - \log_e(2))\int_a^b\int_0^{2\pi}\int_{\varepsilon}^{1-\varepsilon} x\cos\vartheta dxd\vartheta dz' = 0$$

So that

$$A_{\varphi}(r,z) = \frac{\mu_0 j}{4\pi}\int_a^b\int_0^{2\pi}\int_{\varepsilon}^{1-\varepsilon}\{2\gamma\delta\beta - 2\gamma\ln\delta\beta\}dxd\vartheta dz' + K_0 + K_2 + O(\alpha^2).$$

XII. Considering The Order $\delta\beta$ And γ Terms In The Expression For $A_{\varphi}(r,z)$.

Considering the $O(\delta\beta)$ and $O(\gamma)$ terms and denoting this integral as

$$\Delta_1 = \frac{\mu_0 j}{2\pi}(1-2\varepsilon)\int_a^b\int_0^{2\pi}(\delta x\cos\vartheta(x^2+r^2-2xr\cos\vartheta)^{1/2} - \Gamma x\cos\vartheta)dxd\vartheta$$

Where $\Gamma = \log_e|\delta\beta|$. With slight manipulation it can be shown that

$$\Delta_1 = 4\frac{\mu_0 j}{\pi}(1-2\varepsilon)\delta\int_a^b x(x+r)dx\int_0^{\pi/2}\sin^2 u(1-\lambda^2\sin^2 u)^{1/2}du - 2\frac{\mu_0 j}{\pi}(1-2\varepsilon)\delta\int_a^b x(x+r)dx\int_0^{\pi/2}(1-\lambda^2\sin^2 u)^{1/2}du$$

Where

$$\lambda^2 = \frac{2k^2}{1+k^2}, k^2 = \frac{\eta}{\mu^2}, \mu^2 = x^2+r^2, \eta = 2xr, \frac{\vartheta}{2} = \frac{\pi}{2} - u$$

It can be shown that (see Gradsteyn and Ryzhik [7])

$$\int_0^{\pi/2}\sin^m u\cos^n u(1-k^2\sin^2 u)^{1/2}du = \frac{1}{2}B\left(\frac{m+1}{2},\frac{n+1}{2}\right)F\left(\frac{m+1}{2},-\frac{1}{2},\frac{m+n+2}{2},k^2\right)$$

For $m>-1, n>-1, |k^2|<1$, where $B(p,q)$ is the Beta function and $F(a,b,c,z^2)$ is the hypergeometric function whose convergence has already been discussed, thus Δ_1 can easily be evaluated. Now the term containing the logarithm of β must be considered, denoting this integral as Δ_2 then

$$\Delta_2 = -\frac{\mu_0 j}{4\pi}(1-2\varepsilon)\int_a^b xdx\int_0^{2\pi}\cos\vartheta(x^2+r^2-2xr\cos\vartheta)d\vartheta$$

Once again this integral has be computed see Pavlika [10], thus finally

$$A_{\varphi}(r,z) = K_0 + K_1 + \Delta_1 + \Delta_2 + O(\alpha^2)$$

Where K_0, K_2, Δ_1 and Δ_2 are now known.

XIII. Conclusions

The two methods of solution were found to be in good agreement however more terms are required for the method of solution based on the Euler-Maclaurin summation formula. The summations were performed from -200 to 200 with a change only in the fourth decimal place occurring when the number of terms in the summation was doubled. The effect of the permeability of the iron is shown in figures 2, 3, 4 and 5.

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XIV. TABLES

Table 1.
Values of $A_\phi(r,z)$ using the Elliptic Integrals of the 1st and 2nd kind, accurate $O(\delta^8)$.

r	Z	μ=10³	μ=10²	μ=10	μ=1
0	0.1	0	0	0	0
0.1	0.1	0.89172	0.881238	0.7576	0.3481
0.2	0.1	1.79492	1.762867	1.5141	0.6902
0.3	0.1	2.69390	2.645277	2.2679	1.0201
0.4	0.1	3.59466	3.528858	3.0178	1.3319
0.5	0.1	4.49780	4.414002	3.7625	1.6196
0.1	0.2	0.89782	0.882508	0.7642	0.3733
0.1	0.3	0.89596	0.883737	0.7693	0.3926
0.1	0.4	0.89920	0.884629	0.7726	0.4049
0.1	0.5	0.89943	0.884955	0.7738	0.4091

Table 2.
Values of $B_r(r,z)$ using the Elliptic Integrals of the 1st and 2nd kind, accurate $O(\delta^8)$.

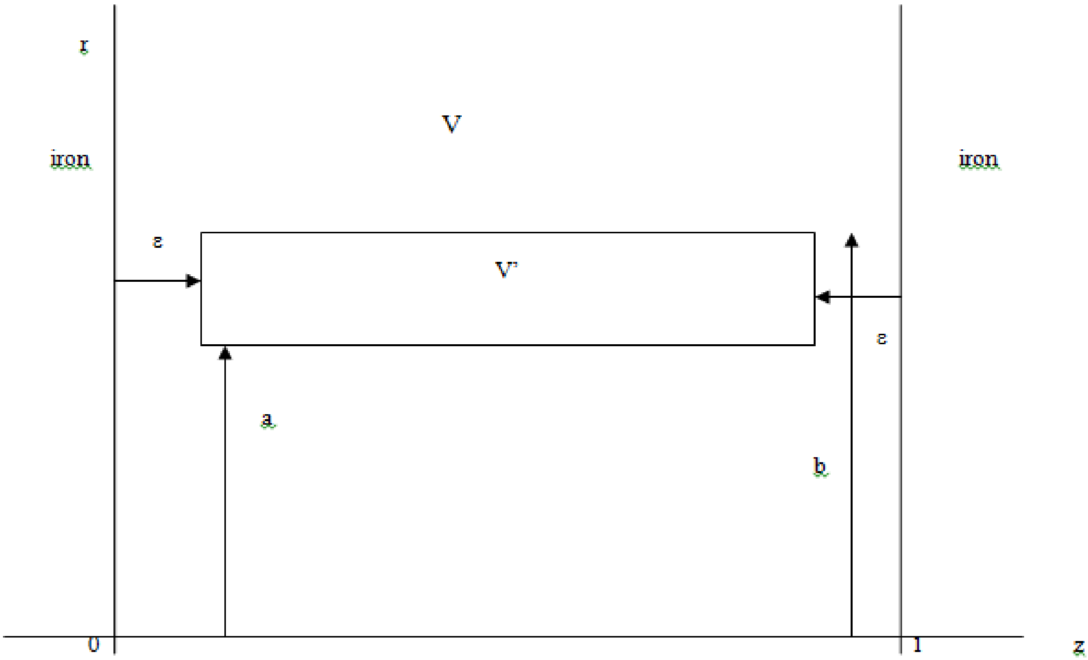
r	z	μ=10³	μ=10²	μ=10	μ=1
0.1	0.1	5.832E-3	0.0163	0.1042	0.0362
0.2	0.1	1.315E-2	0.0343	0.2120	0.0776
0.3	0.1	2.344E-2	0.0556	0.3674	0.1426
0.4	0.1	3.819E-2	0.0820	0.4521	0.1599
0.5	0.1	5.887E-2	0.1151	0.5914	2.0972
0.1	0.2	8.426E-3	0.0166	0.0852	0.2937
0.1	0.3	8.083E-3	0.0136	0.0607	0.2072
0.1	0.4	4.898E-3	0.0071	0.0316	0.0107
0.1	0.5	0	0	0	0

Table 3.
Values of $B_z(r,z)$ using the Maclaurin Series Expansion accurate $O(r^8)$.

r	Z	μ=10³	μ=10²	μ=1
0	0.1	17.9170	17.6164	6.9822
0.1	0.1	17.0150	17.6151	7.0023
0.2	0.1	17.9091	17.6112	7.0628
0.3	0.1	17.8991	17.6047	7.1635
0.4	0.1	17.8852	17.5965	7.3046
0.5	0.1	17.8673	17.5839	7.4860
0.1	0.2	17.9732	17.6546	7.5233
0.1	0.3	17.9723	17.6771	7.9259
0.1	0.4	17.9861	17.6996	8.1803
0.1	0.5	17.9867	17.7015	8.2673

XV. FIGURES

Figure 1. A toroidal conductor V' of rectangular cross section located midway between two semi infinite regions of iron of finite permeability. The region V is assumed to be insulating.



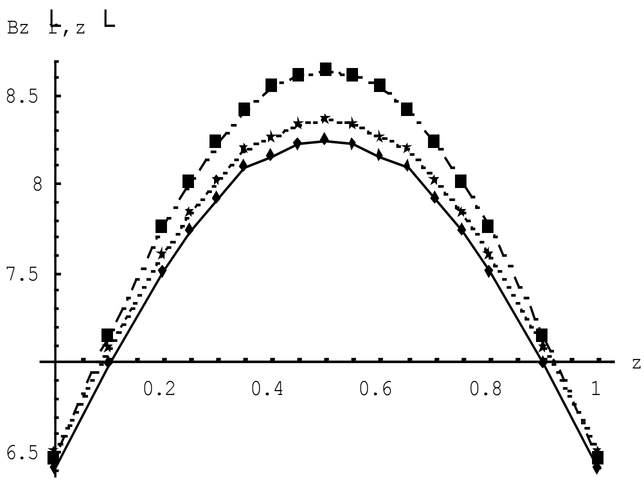


Figure 2. The variation of $B_z(r,z)$ with *r* and *z* for two semi-infinite regions of iron of unit permeability. *+:r=0.3, ™:r=0.2, •:r=0.1*

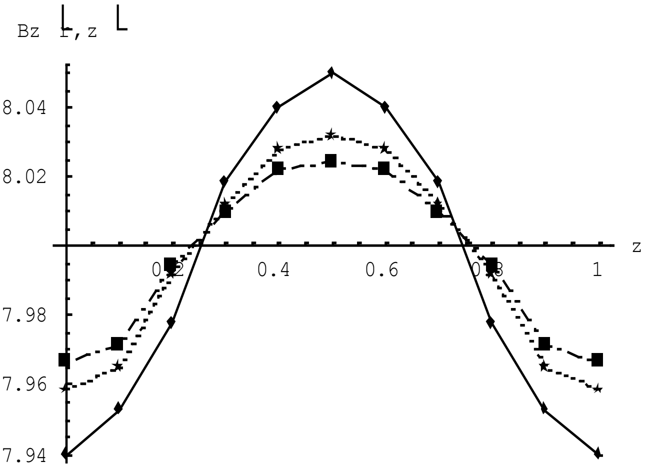


Figure 3. The variation of $B_z(r,z)$ with *r* and *z* for two semi-infinite regions of iron of infinite permeability. *+:r=0.1, ™:r=0.2, •:r=0.3*

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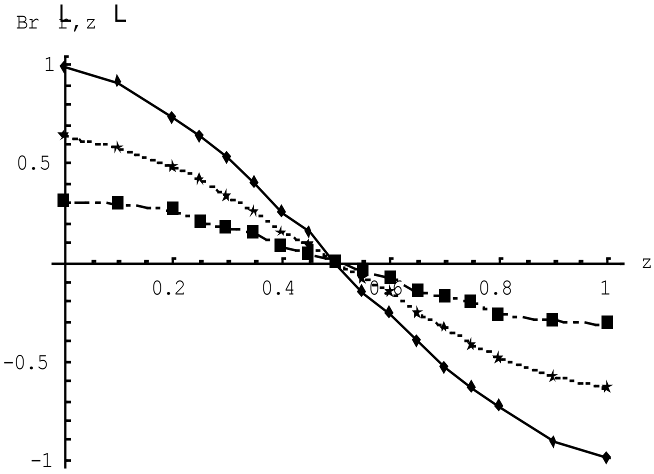

Figure 4. The variation of $B_r(r,z)$ with *r* and *z* for two semi-infinite regions of iron of unit permeability. *+:r=0.1, ™:r=0.2,*

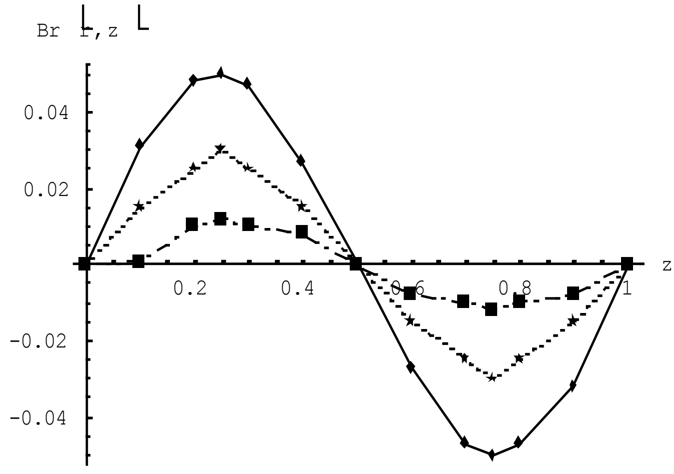

Figure 5. The variation of $B_r(r,z)$ with *r* and *z* for two semi-infinite regions of iron of infinite permeability. *+:r=0.1, ™:r=0.2, •:r=0.3*