

# Resonance in the Motion of Mars Orbiter Due to Linear Drag

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**Abstract--** This paper is to discuss the effects of linear drag on the resonant motion of Mars orbiter. In presence of linear drag three resonances 1:1, 2:1, 3:1 occur. Also discuss the amplitude and time period of the orbiter at all these resonant points.

**Keywords--**Resonance, Liner drag, Mars orbiter. Amplitude and Time period.

## I. INTRODUCTION

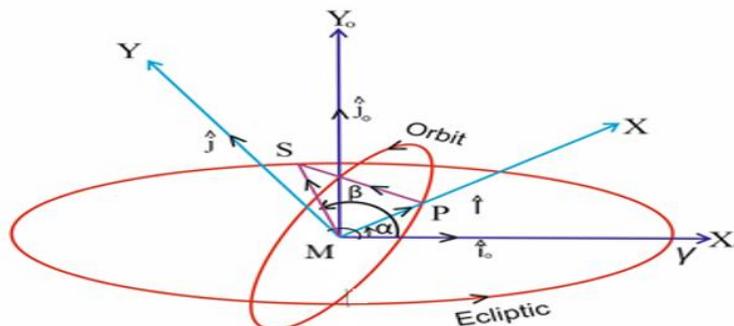
Resonant motion of celestial bodies has a significant role in solar dynamics. Furthermore, the dissipation mechanism is also multi-dimensional in solar dynamics. The non-gravitational dissipative force arises because of gas nebula moving in solar system is named by the term “Stoke’s drag” which is proportional to the particle’s velocity pertaining to gas and valid for low Renold number  $<10$ . The specific case of Stoke’s drag is called linear drag.

Problems of resonance play a crucial role in solar dynamics during solving the equations of motion. Resonance may be manifested as the appearance of small divisors in the solutions of the equations of motion. Orbital resonance of earth satellite with respect to lunisolar gravity and direct solar radiation pressure with particular reference to those resonances, the occurrence of which is dependent only on the satellite’s orbital inclination was studied by Hughes in (1980). Weideschilling et al (1985) discovered the behaviour of resonance trapping in a gas rich scenario. Patterson (1987) extended the work of Weidenschilling to resonances of any order and showed how planetary embryos could have formed at two-body external resonances by accretion of infinitesimals caught in these orbits.

Bhatnagar et al (1986) examined the motion of a satellite by taking gravitational forces of the moon, earth and the sun (with radiation pressure). Ferraz-Mello (1992) studied “averaging of the elliptic asteroidal problem with Stokes drag” and with the assistance of Beague and Ferraz-Mello (1993) studied resonance trapping Stokes drag dissipation in the primordial solar nebula. Celletti and Stefanelli (2011) have shown that the semi major axis decreases due to dissipation and consequently the collision takes place between one of the primaries and minor bodies. Quarles et al (2012) has studied the resonances for coplanar Circular Restricted Three-Body Problem for the mass ratio between 0.10 and 0.15 and used the method of maximum Lyaponav exponent to locate the resonances. They showed that for high value of resonance, orbital stability is ensured where single resonance is present.

Sushil et al (2013) worked on resonance in a geocentric satellite due to earth’s equatorial ellipticity. Rosemary (2013) has given detailed description of the perturbation theory to determine the location of resonance based on approximations to a harmonic oscillation. Kaul et al (2018) studied resonance in the motion of geocentric satellite due to PR-drag. Further they have discussed the resonance in the motion of geocentric satellite due to PR-drag and equatorial ellipticity of the earth. Hassan et al (2023) has studied the effect of Stokes drag on the resonant motion of a geocentric satellite. Presently we have proposed to study the effect of linear drag on the resonant motion of a Mars orbiter.

## II. THE EQUATIONS OF MOTION



Let us considering the inertial frame  $(M, X_0Y_0Z)$  whose origin at the Mars  $M$  and a rotating frame  $(M, XYZ)$  relative to the inertial one, where  $\overline{MX_0}$  passes through the vernal equinox  $\mathcal{V}$ . Let  $\hat{i}_0, \hat{j}_0$  and  $\hat{i}, \hat{j}$  be the unit vectors along the axes of inertial frame and rotating frame with

common unit vector  $\hat{k}$  along the vertical axis MZ (not seen in the figure). Let  $\overline{MP} = \vec{r}$  be the position vector of the orbiter P,  $\overline{SP} = \vec{\rho}$  be the position of the Sun S relative to the mars M and  $\overline{SM} = \vec{R}$ . If  $\mu, \sigma$  and  $m$  be the masses of the Sun, Mars and the orbiter respectively then their mutual gravitational forces are given by

$$\vec{F}_{MP} = -\frac{G\sigma m}{r^3} \vec{r}, \quad \vec{F}_{SP} = -\frac{Gm\mu}{\rho^3} \vec{\rho}, \quad \vec{F}_{SM} = -\frac{G\mu\sigma}{R^3} \vec{R} \quad (1)$$

The force of linear drag applied on the orbiter P is given by

$$\vec{L} = -cm(\dot{\vec{\rho}} - \vec{\rho} \times \hat{k}) \quad (2)$$

Where  $c \in [0, 1)$  is the dissipative constant [(Beauge and Ferraz-Mello (1993)]. Let  $\vec{\omega}$  be angular velocity of

the rotating frame relative to the inertial frame and  $\hat{i}$  the unit vector along the direction of orbiter then the equation of motion of orbiter in rotating frame can be written as

$$\ddot{\vec{r}} = \frac{\partial^2 r}{\partial t^2} \hat{i} + 2 \frac{\partial r}{\partial t} (\omega \times \hat{i}) + r \left( \frac{\partial \vec{\omega}}{\partial t} \times \hat{i} \right) + r \left[ (\vec{\omega} \square \hat{i}) - (\vec{\omega} \square \vec{\omega}) \hat{i} \right] \quad (3)$$

Let  $\alpha$  be the angle of direction of the orbiter with the direction of vernal equinox, then  $\vec{\omega} = \dot{\alpha} \vec{k}$ , where  $\dot{\alpha}$  is the angular velocity of the orbiter. Thus, the equation (3) reduced to

$$\ddot{\vec{r}} = \left( \frac{\partial^2 r}{\partial t^2} - r \dot{\alpha}^2 \right) \hat{i} + \left( 2 \dot{\alpha} \frac{\partial r}{\partial t} + r \ddot{\alpha} \right) \hat{j} \quad (4)$$

In the triangle MPS

$$\ddot{\vec{r}} = \ddot{\vec{\rho}} - \ddot{\vec{R}} = \frac{\vec{F}_{SP} + \vec{F}_{MP} + \vec{L}}{m} - \frac{\vec{F}_{SM}}{\sigma} = -\frac{G\mu}{\rho^3} \vec{\rho} - \frac{G\sigma}{r^3} \vec{r} - c(\dot{\vec{\rho}} - \vec{\rho} \times \hat{k}) + \frac{G\mu}{R^3} \vec{R}.$$

If  $\hat{R}$  be the unit vector along  $\vec{R}$  and  $\beta$  be the angle of the direction of the sun with the direction of vernal equinox  $\mathcal{V}$  then  $\hat{R} = \cos \beta \hat{i}_0 + \sin \beta \hat{j}_0$  and  $\dot{\beta}^2 = \frac{G\mu}{R^3}$  implies that

$$\ddot{\vec{r}} = -\frac{G\sigma}{r^2} \hat{i} + \dot{\beta}^2 R (\cos \beta \hat{i}_0 + \sin \beta \hat{j}_0) - c(\dot{\vec{\rho}} - \vec{\rho} \times \hat{k}) - \frac{G\mu}{\rho^3} (r \hat{i} + R \cos \beta \hat{i}_0 + R \sin \beta \hat{j}_0). \quad (5)$$

Scalar product of  $\hat{i}$  with (4) and (5) and that of  $\hat{j}$  with (4) and (5), comparing the results one can find the equations of motions of the orbiter in polar form as

$$\frac{\partial^2 r}{\partial t^2} - \dot{\alpha}^2 r + \frac{G\sigma}{r^2} = R \left( \dot{\beta}^2 - \frac{G\mu}{\rho^3} \right) \cos(\alpha - \beta) - c(\dot{\vec{\rho}} \square \hat{i} - (\vec{\rho} \times \hat{k}) \square \hat{i}) - \frac{G\mu r}{\rho^3} \quad (6)$$

$$\frac{\partial}{\partial t}(r^2\dot{\alpha}) = -\dot{\beta}^2 Rr \sin(\alpha - \beta) - cr \left( \dot{\vec{p}} \cdot \hat{j} - (\vec{p} \times \hat{k}) \cdot \hat{j} \right) - \frac{GR\mu r}{\rho^3} \sin(\alpha - \beta) \quad (7)$$

These equations are not integrable, so we replace  $r$  and  $\dot{\alpha}$  by their steady state value  $r_0$  and  $\dot{\alpha}_0$  by perturbation technique which can be introduced in (6) and (7) as  $\alpha = \dot{\alpha}_0 t$  &  $\beta = \dot{\beta} t$

$$\frac{d^2 r}{dt^2} - \dot{\alpha}^2 r + \frac{G\sigma}{r^2} = R \left( \dot{\beta}^2 - \frac{G\mu}{\rho^3} \right) \cos(\dot{\alpha}_0 - \dot{\beta})t - c \left( \dot{\vec{p}} \cdot \hat{i} - (\vec{p} \times \hat{k}) \cdot \hat{i} \right) - \frac{G\mu r_0}{\rho^3} \quad (8)$$

$$\frac{d}{dt}(r^2\dot{\alpha}) = -\dot{\beta}^2 Rr_0 \sin(\dot{\alpha}_0 - \dot{\beta})t - cr_0 \left( \dot{\vec{p}} \cdot \hat{j} - (\vec{p} \times \hat{k}) \cdot \hat{j} \right) - \frac{GR\mu r_0}{\rho^3} \sin(\dot{\alpha}_0 - \dot{\beta})t \quad (9)$$

At steady state

$$\dot{\vec{p}} = R\dot{\beta} \sin(\dot{\alpha}_0 - \dot{\beta}_0)t \hat{i} + R\dot{\beta} \cos(\dot{\alpha}_0 - \dot{\beta}_0)t \hat{j} + \dot{\alpha}_0 r_0 \hat{j}$$

For central orbit of orbiter  $r^2\dot{\alpha} = \text{constant} = h$  (say) and  $r = 1/u$  in (8) we get

$$\frac{d^2 u}{d\alpha^2} + u = \frac{G\sigma}{r_0^4 \dot{\alpha}_0^2} + \frac{R}{\dot{\alpha}_0^2} \left( -\dot{\beta}^2 + \frac{G\mu}{\rho^3} \right) u^2 \cos(\dot{\alpha}_0 - \dot{\beta})t + \frac{cRu^2}{\dot{\alpha}_0^2} (\dot{\beta} + 1) \sin(\dot{\alpha}_0 - \dot{\beta})t + \frac{G\mu r_0 u^2}{\rho^3 \dot{\alpha}_0^2} \quad (10)$$

### III. RESONANCE IN THE MOTION OF THE MARS ORBITER IN PRESENCE OF LINEAR DRAG:

The complete solution of the unperturbed equation of motion  $\frac{d^2 u}{d\alpha^2} + u = \frac{G\sigma}{r_0^4 \dot{\alpha}_0^2}$  is given by

$$\frac{l}{r} = 1 + e \cos(\alpha - \psi), \text{ where } l = a(1 - e^2) \text{ and } e, \psi \text{ are constants. Thus } u = \frac{1 + e \cos(\alpha - \psi)}{a(1 - e^2)}.$$

Let us consider  $\alpha - \psi = \dot{\alpha}_0 t = nt$  (say) where  $n$  be the frequency of the satellite.

Since eccentricity  $e < 1$ , so  $(1 + e \cos nt)^h \approx 1 + h_1 e \cos nt$ .

Hence by using  $n$  and  $\frac{d^2 u}{dt^2} = \dot{\alpha}_0^2 \frac{d^2 u}{d\alpha^2}$  in equation (10), then we get the perturbed equation of the motion of the orbiter is

$$\begin{aligned} \frac{d^2 u}{dt^2} + n^2 u = & L_1 + L_2 \cos nt + L_3 \cos \dot{\beta} t + L_4 \sin \dot{\beta} t + L_5 \cos(n - \dot{\beta})t \\ & + L_6 \sin(n - \dot{\beta})t + L_7 \cos(2n - \dot{\beta})t + L_8 \sin(2n - \dot{\beta})t \end{aligned} \quad (11)$$

Where,

$$\begin{aligned}
 L_1 &= \frac{G\sigma}{r_0^4} + \frac{G\mu r_0}{\rho^3 a^2 (1-e^2)^2}, & L_2 &= \frac{2G\mu r_0 e}{\rho^3 a^2 (1-e^2)^2}, & L_3 &= \frac{Re\left(\frac{G\mu}{\rho^3} - \dot{\beta}^2\right)}{a^2 (1-e^2)^2} \\
 L_4 &= \frac{-cR(\dot{\beta}+1)e}{a^2 (1-e^2)^2}, & L_5 &= \frac{R\left(\frac{GM}{\rho^3} - \dot{\beta}^2\right)}{a^2 (1-e^2)^2}, & L_6 &= \frac{cR(\dot{\beta}+1)}{a^2 (1-e^2)^2} \\
 L_7 &= \frac{Re\left(\frac{G\mu}{\rho^3} - \dot{\beta}^2\right)}{a^2 (1-e^2)^2} = L_3, & L_8 &= \frac{cR(\dot{\beta}+1)e}{a^2 (1-e^2)^2} = -L_4
 \end{aligned}$$

The solution of equation (11) is given by

$$\begin{aligned}
 u &= L \cos(nt - \varepsilon) + \frac{L_1}{n^2} + \frac{L_2 t \sin nt}{2n} + \frac{L_3 \cos \dot{\beta} t}{n^2 - \dot{\beta}^2} + \frac{L_4 \sin \dot{\beta} t}{n^2 - \dot{\beta}^2} + \frac{L_5 \cos(n - \dot{\beta})t}{n^2 - (n - \dot{\beta})^2} \\
 &+ \frac{L_6 \sin(n - \dot{\beta})t}{n^2 - (n - \dot{\beta})^2} + \frac{L_7 \cos(2n - \dot{\beta})}{n^2 - (2n - \dot{\beta})^2} + \frac{L_8 \sin(2n - \dot{\beta})t}{n^2 - (2n - \dot{\beta})^2}
 \end{aligned} \tag{12}$$

Where  $\varepsilon$  is constant of integration. On vanishing the denominator of any term of equation (12) we get some points at which motion becomes indeterminate and hence resonance occurs at these points. Thus, the resonances occur at the points  $n = \dot{\beta}$ ,  $2n = \dot{\beta}$  and  $3n = \dot{\beta}$ . All the three resonances 1:1, 2:1 and 3:1 occur due to linear drag.

#### IV. GENERALIZATION OF AMPLITUDE AND TIME PERIOD

By using Brown and shook (1933) and Hassan et.al (2022) the generalization formula of the amplitude A and the time period T at the resonant point  $m_1 n = m_2 \dot{\beta}$  where  $m_1, m_2 \in N$  for the equation is of the form  $\frac{d^2 u}{dt^2} + n^2 u = L_s \phi$  where  $s \in N$  are

$$A = \frac{\sqrt{2} \cos \frac{m_1}{m_2} \beta}{\sqrt{|L_s|} n_0} \quad \text{and} \quad T = \frac{2\sqrt{2}\pi \cos \frac{m_1}{m_2} \beta}{\sqrt{|L_s|} n_0} \tag{13}$$

Using the result of (13) we get the amplitudes  $A_1, A_2, A_3$  and time periods  $T_1, T_2, T_3$  at the resonant points  $n = \dot{\beta}$ ,  $2n = \dot{\beta}$  and  $3n = \dot{\beta}$  respectively.

Where

$$A_1 = \frac{\sqrt{2}a(1-e^2)\cos\beta}{\sqrt{cR(\dot{\beta}+1)}en_0}$$

$$A_2 = \frac{\sqrt{2}a(1-e^2)}{\sqrt{cR(\dot{\beta}+1)}}\cos\frac{\beta}{2}$$

$$A_3 = \frac{\sqrt{2}a(1-e^2)}{\sqrt{cR(\dot{\beta}+1)}en_0}\cos\frac{\beta}{3}$$

$$T_1 = \frac{2\sqrt{2}\pi a(1-e^2)\cos\beta}{\sqrt{cR(\dot{\beta}+1)}en_0}$$

$$T_2 = \frac{2\sqrt{2}\pi a(1-e^2)}{\sqrt{cR(\dot{\beta}+1)}}\cos\frac{\beta}{2}$$

$$T_3 = \frac{2\sqrt{2}\pi a(1-e^2)}{\sqrt{cR(\dot{\beta}+1)}en_0}\cos\frac{\beta}{3}$$

#### V. CONCLUSION

In section 1, of this manuscript, the previous works have been cited. In section 2, the polar equations of motion of the Mars orbiter have been established in presence of linear drag in rotating frame relative to the mars. To reduce the chances of non-integrability of the equations of motion, we used perturbation technique by taking the steady state values of the position vector and angular velocity of orbiter. In section 3, we have solved first the unperturbed equation of motion. The solution of perturbed equation (11) of motion in equation (12). By making denominator of any term from 4<sup>th</sup> to 9<sup>th</sup> to zero  $u$  becomes infinity and hence the motion of the orbiter becomes indeterminate. Thus  $n = \dot{\beta}$ ,  $2n = \dot{\beta}$  and  $3n = \dot{\beta}$  are three resonances of the problem all of them are occurred due to linear drag. In section 4, we have found the amplitudes and time periods at all the resonant points which are occurred due to linear drag.

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