International Journal of Recent Development in Engineering and Technology
Website: www.ijrdet.com (ISSN 2347-6435(Online) Volume 14, Issue 12, December 2025)

Derivation of Channel Multipoles in the Linearized Theory of Water Waves

Nityananda Thakur
Department of Mathematics, Surendranath Evening College, 24/2 M.G. Road, Kolkata-700009, India
corresponding author: e-mail: nitya.iitm@gmail.com

***Abstract* - The method presented here is new in the context of channel scattering problems although the basis of the idea is familiar in the water waves. Channel multipoles are derived in the present paper. The method of constructing the channel multipoles is presented here in a systematic manner for deep water with ice-cover surface. When the ice-cover condition reduces to free surface condition, the channel multipoles exactly coincide with the channel multipoles for water with free surface.**

***Key words* - Channel multipoles, Helmholtz equation, multipole potentials, free surface, ice-cover, Laplace operator.**

I. Introduction

Problem concerning the hydrodynamic properties of a vertical circular cylinder, immersed through the free surface and extending part way to the bottom, situated in the centre of a channel have been studied using linear water wave theory (cf. Yeung and Sphaier (1989)). One reason for interest in this problem is because of the need to know how the side walls of a wave tank affect the results of experiments on a relatively large model. Many of the problems associated with the solution of such problems can be explained with reference to much simpler problems. A particular, and not unrealistic, geometry that has received considerable attention in recent years is that of a vertical circular cylinder, and linear radiation and diffraction problems relating to such an obstacle are now well understood, (cf. Linton and Evans (1992)). McIver and Bennet (1993) investigated the scattering of water waves by axisymmetric bodies in a channel by using the method of multipole for constructing the channel multipoles.

The method presented here is new in the context of channel scattering problems al- though the basis of the idea is familiar in the water waves. For example, Ursell (1950) solved the two-dimensional problem of scattering by a submerged, circular cylinder by constructing the solution from an infinite set of multipole potentials. Each multipole individually satisfied all the conditions of the problems except the boundary condition on the cylinder surface.

Ursell expressed the solution as a sum over the multipoles, with unknown coefficients, and imposed the final boundary condition to give an infinite system for those coefficients which is readily solved by truncation. Subsequently, Thorne (1953) derived a range of multipoles in two and three-dimensions that allow the straight- forward solution of many problems. Following the procedure used by Thorne, 'Channel multipoles' are derived in the present paper. Each multipole is a singular solution of the two-dimensional Helmholtz equation that satisfies the boundary condition of no flow through the walls and the radiation conditions of outgoing waves at large distances along the channel. A restricted set of these multipoles has already been used by Callan, Linton and Evans (1991) and McIver (1991) in constructing solutions for waves trapped by a vertical cylinder on the centre line of a channel.

Recently there is a considerable interest in the mathematical investigation of ice-wave interaction problems due to an increase in the scientific activities in polar oceans. Instead of a free surface, a polar ocean in covered by ice. The ice-cover is modelled as a thin uniform sheet of ice of which still a smaller part is immersed in water, and is composed of materials having elastic properties. Already, quite a number researchers have considered various types of water wave problems in a polar ocean with an ice-cover modelled as a thin elastic plate.

The method of constructing the channel multipoles is presented here in a systematic manner for deep water with ice-cover surface. When the ice-cover condition reduces to free surface condition, the channel multipoles exactly coincide with the channel multipoles for deep water with free surface (cf. McIver and Bennet (1993)).

II. Formulation Of The Problem

An infinite long channel of uniform depth h has parallel walls a distance 2b apart. A cartesian coordinates (x, y, z) are chosen with the origin in the mean free surface and midway between the channel walls such that x-axis is directed along the channel, $(-\infty < x < \infty)$ and the y-axis being measured vertically

International Journal of Recent Development in Engineering and Technology
Website: www.ijrdet.com (ISSN 2347-6435(Online) Volume 14, Issue 12, December 2025)

upwards. A vertical circular cylinder of radius a extends throughout the depth and has its axis at $(x, z) = (0, d)$. The relationship between polar coordinates (r, θ) and cartesian coordinates is given by the equations

$$x = r\cos\theta \quad and \quad z - d = r\sin\theta. \qquad (2.1)$$

A plane incident wave approaches the vertical circular cylinder along the x-axis from $x = -\infty$. The incident wave parameters are: A, wave amplitude; ω, wave frequency. With the usual assumptions of an inviscid, incompressible fluid there exists a harmonic velocity potential $\Phi(x, y, z, t)$. It is further assumed that all motion is time-harmonic and so assuming the linear water wave theory of irrotational surface wave may be described by a velocity potential

$$\Phi(x, y, z, t) = \Re\left\{-\frac{igA\cosh\lambda(y+h)}{\omega\cosh\lambda h}\,\phi_T(x, z)e^{i\omega t}\right\}, \qquad (2.2)$$

where λ is the unique positive real root of the dispersion equation

$$\lambda(D\lambda^4 + 1 - \epsilon K)\sinh\lambda h - K\cosh\lambda h = 0, \qquad (2.3)$$

where $\Re$ indicates that the real part is to be taken, g is gravitational acceleration and $\phi_T(x, z)$ is a complex valued potential function, $K = \frac{\omega^2}{g}$, $D = \frac{Eh_0^3}{12(1-\nu^2)\rho_I g}$ and $\epsilon = \frac{\rho_0}{\rho_I}h_0$, ρ_0 is the density of ice, ρ_I is the density of water, h_0 is the small thickness of the ice-cover and E, ν are the Young's modulus and Poisson's ratio of the ice. The form of the potential has been chosen to satisfy the linearized ice-cover condition

$$D\left(\frac{\partial^4}{\partial x^4} + 1 - \epsilon K\right)\phi_y + K\phi = 0 \quad on \quad y = 0 \qquad (2.4)$$

and the bottom condition

$$\frac{\partial\phi}{\partial y} = 0 \quad on \quad y = -h \qquad (2.5)$$

The velocity potential $\Phi(x, y, z, t)$ must satisfy

$$\nabla^2\Phi = 0 \qquad (2.6)$$

where ∇^2 denotes the three-dimensional Laplace operator.

On substituting the form (2.2), the complex-valued function $\phi_T(x, z)$ may be seen to satisfy the Helmholtz equation

$$(\nabla^2 - \gamma^2)\,\phi_T = 0 \qquad (2.7)$$

where γ is the wave number component along the z-direction.

The incident wave is described by

$$\phi_I = e^{i\lambda x} = e^{i\lambda r\cos\theta} = \sum_{n=0}^{\infty}\epsilon_n i^n J_n(\lambda r)\cos n\theta \qquad (2.8)$$

(Abramowitz and Stegun (1965)) where

$$\epsilon_0 = 1, \ \epsilon_n = 2 \ for \ n \geq 1 \qquad (2.9)$$

and J_n is the Bessel function of first kind.

The total potential ϕ_T can be decomposed into two parts as follows:

$$\phi_T = \phi_I + \phi \qquad (2.10)$$

The potential ϕ describing the scattered wave field will then satisfy the Helmholtz equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} + \lambda^2\phi = 0, \quad \text{in the fluid region} \qquad (2.11)$$

The conditions of no flow through the channel walls

$$\frac{\partial\phi}{\partial z} = 0 \quad on \quad z = \pm b \qquad (2.12)$$

and the cylinder surface,

$$\frac{\partial\phi}{\partial r} = -\frac{\partial\phi_I}{\partial r} \quad on \quad r = a, \qquad (2.13)$$

and the radiation condition specifying that the scattered waves behave as outgoing as $|x| \to \infty$.

The solution for ϕ is expressed in terms of channel multipoles, derived in the following section.

III. Derivation Of Channel Multipoles

The channel multipoles derived here are singular solutions of the Helmholtz equation (2.11) that, by construction, satisfy the conditions (2.12) on the channel wall and the radiation condition. Each multipole is singular at the point $(x, z) = (0, d)$, that is at $r = 0$ where r is a polar coordinate as defined in (2.1). Two sets of multipoles are defined. The first, denoted by ϕ_n, have a singular part $H_n(\lambda r)\cos n\theta$ and the second, denoted by ψ_n, have a singular part $H_n(\lambda r)\sin n\theta$. Here, H_n denotes the Hankel function of the first kind and order n.

The basis of the derivation is given by the integral representation

$$H_n(\lambda r)e^{in\theta} = \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{e^{-i\lambda xt - \gamma\lambda(z-d)}}{\gamma}e^{in\sin^{-1}t}dt, \quad z > d, \qquad (3.1)$$

where $\gamma = (t^2 - 1)^{1/2} = -i(1 - t^2)^{1/2}$ (cf. Twersky (1962)) after a rotation of the axes through $\pi/2$. The path of integration runs beneath the branch point at $t = 1$ and above that at $t = -1$. Define τ by

International Journal of Recent Development in Engineering and Technology
Website: www.ijrdet.com (ISSN 2347-6435(Online) Volume 14, Issue 12, December 2025)

$$\cosh\tau = t,\quad \sinh\tau = \gamma \qquad (3.2)$$

By combining (3.1) with a similar expression where n is replaced by $-n$ and considering the cases of odd and even n separately, it can be shown that

$$H_n(\lambda r)\cos n\theta = \frac{i^{n-1}}{\pi}\int_{-\infty}^{\infty}\frac{e^{-i\lambda xt-\gamma\lambda|z-d|}}{\gamma}\cosh n\tau\,dt, \qquad (3.3)$$

and

$$H_n(\lambda r)\sin n\theta = -sgn(z-d) \times \frac{i^{n}}{\pi}\int_{-\infty}^{\infty}\frac{e^{-i\lambda xt-\gamma\lambda|z-d|}}{\gamma}\sinh n\tau\,dt, \qquad (3.4)$$

where the results

$$\cos(2n\ \sin^{-1}t) = (-1)^n\cosh 2n\tau,$$
$$\cos\big((2n+1)\sin^{-1}t\big) = i(-1)^n\sinh(2n+1)\tau,$$
$$\sin(2n\sin^{-1}t) = -i(-1)^n\sinh 2n\tau,$$
$$\sin\big((2n+1)\sin^{-1}t\big) = (-1)^n\cosh(2n+1)\tau, \qquad (3.5)$$

have been used.

Using the method of Thorne (1953) channel multipoles can be constructed. For the multipoles ϕ_n corresponding to (3.3) we write

$$\phi_n = H_n(\lambda r)\cos n\theta + \frac{i^{n-1}}{\pi}\int_{-\infty}^{\infty}\left(A(t)e^{\gamma\lambda(z-d)} + B(t)e^{-\gamma\lambda(z-d)}\right)\frac{e^{-i\lambda xt}}{\gamma}\cosh n\tau\,dt \qquad (3.6)$$

where $A(t)$ and $B(t)$ are functions of t to be determined such that the integral exists in some sense. The channel multipoles satisfy the boundary conditions of no flow through the channel walls at $z = \pm b$. Then the unknown constants are obtained as

$$A(t) = \frac{e^{-2\gamma\mu}+e^{2\gamma\nu}}{2\sinh 2\gamma\mu}$$

and

$$B(t) = \frac{e^{-2\gamma\mu}+e^{-2\gamma\nu}}{2\sinh 2\gamma\mu}$$

where $\mu = \lambda b$ and $\nu = \lambda d$. Thus, we obtain

$$\phi_n = H_n(\lambda r)\cos n\theta + \frac{i^{n-1}}{2\pi}\int_{-\infty}^{\infty}\frac{e^{\gamma\lambda(z-d)}(e^{-2\gamma\mu}+e^{2\gamma\nu})+e^{-\gamma\lambda(z-d)}(e^{-2\gamma\mu}+e^{-2\gamma\nu})}{\gamma\sinh 2\gamma\mu} \times e^{-i\lambda xt}\cosh n\tau\,dt. \qquad (3.7)$$

Using the integral representation (3.3) into (3.7), then the equation (3.7) can be written as

$$\phi_n = \frac{i^{n-1}}{\pi}\int_{-\infty}^{\infty}\frac{\cosh\gamma(\lambda|y-d|-2\mu)+\cosh\gamma\lambda(y+d)}{\gamma\sinh 2\gamma\mu} \times e^{-i\lambda xt}\cosh n\tau\,dt. \qquad (3.8)$$

In general, the integrals in equations (3.7) and (3.8) have poles at the solution of $2\gamma\mu = \pm im\pi$, where m is a positive integer. The corresponding values of t are $\pm t_m$ where

$$t_m = (1-(m\pi/2\mu))^{1/2},\quad m = 0, 1, 2, \ldots, M \qquad (3.9)$$

$$t_m = i((m\pi/2\mu)^2-1)^{1/2},\quad m \geq M+1 \qquad (3.10)$$

and M is the integer satisfying

$$M\pi < 2\pi < (M+1)\pi \qquad (3.11)$$

The $M+1$ poles given by (3.9) lie on the real t-axis and give rise to propagating waves as $|x| \to \infty$. To obtain only outgoing waves the integration path must run beneath the poles on the positive real axis and above the poles on the negative real axis. Applying the residue theorem to evaluate the integral in (3.8) gives the alternative representation

$$\phi_n = \frac{(-i\,sgn\,x)^n}{\mu}\left\{e^{i\lambda|x|} + \sum_{m=1}^{\infty}\left[\cos\left(\frac{m\pi}{2\mu}\lambda(z-d)\right) + (-1)^m\cos\left(\frac{m\pi}{2\mu}\lambda(z+d)\right)\right]t_m^{-1}e^{i\lambda|x|t_m}\cosh n\tau_m\right\}, \qquad (3.12)$$

where

$$\cosh\tau_m = t_m \qquad (3.13)$$

For, $m \geq M+1$, t_m is imaginary and the corresponding terms in the above summation decay exponentially as $|x| \to \infty$ so that, for a given μ, there are only a finite number (≥ 1) of propagating modes. If $\nu = \lambda d = 0$ so that the singular point is on the centre line of the channel. Then only even values of m contribute to the summation and all modes are symmetric in z.

Channel multipoles ψ_n corresponding to (3.4) can be constructed as

$$\psi_n = H_n(\lambda r)\sin n\theta + \frac{i^{n-1}}{\pi}\int_{-\infty}^{\infty}\left(G(t)e^{\gamma\lambda(z-d)} + H(t)e^{-\gamma\lambda(z-d)}\right) \times \frac{e^{-i\lambda xt}}{\gamma}\sinh n\tau\,dt. \qquad (3.14)$$

International Journal of Recent Development in Engineering and Technology
Website: www.ijrdet.com (ISSN 2347-6435(Online) Volume 14, Issue 12, December 2025)

where $G(t)$ and $H(t)$ are unknown functions of t to be found such that the integrals exist in some sense. The channel multipoles satisfy the boundary conditions of no flow through the channel walls at $z = \pm b$. Then the unknown constants are obtained as

$$G(t) = \frac{i}{2}\frac{(e^{-2\gamma\mu} - e^{2\gamma\nu})}{\sinh 2\gamma\mu}$$

and

$$H(t) = -\frac{i}{2}\frac{(e^{-2\gamma\mu} - e^{-2\gamma\nu})}{\sinh 2\gamma\mu}$$

where $\mu = \lambda b$ and $\nu = \lambda d$.

Thus, we obtain

$$\psi_n = H_n(\lambda r)\sin n\theta + \frac{i^n}{2\pi}\int_{-\infty}^{\infty}\frac{e^{\gamma\lambda(z-d)}(e^{-2\gamma\mu}-e^{2\gamma\nu})-e^{-\gamma\lambda(z-d)}(e^{-2\gamma\mu}-e^{-2\gamma\nu})}{\gamma\sinh 2\gamma\mu} \times e^{-i\lambda xt}\sinh n\tau\, dt. \tag{3.15}$$

The location of the poles and the choice of contour is the same as for ϕ_n above with the exception that there are no longer poles at $t = \pm 1$. The expressions corresponding to (3.8) and (3.12) are

$$\psi_n = \frac{i^n}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{sgn}(z-d)\sinh\gamma(\lambda|z-d|-2\mu)-\sinh\gamma\lambda(z+d)}{2\sinh 2\gamma\mu} \times e^{-i\lambda xt}\sinh n\tau\, dt \tag{3.16}$$

and

$$\psi_n = \frac{(-i\,sgn\,x)^{n+1}}{\mu i}\sum_{m=1}^{\infty}\left[\left[\sin\left(\frac{m\pi}{2\mu}\lambda(z-d)\right) + (-1)^m\sin\left(\frac{m\pi}{2\mu}\lambda(z+d)\right)\right] t_m^{-1}e^{i\lambda|x|t_m}\sinh n\tau_m, \tag{3.17}$$

If $\nu = \lambda d = 0$ only the odd values of m contribute to the summation and all the modes are antisymmetric in z. In addition, if $\mu < \frac{\pi}{2}$ then all t_m, with m odd, are pure imaginary and there is no radiation of energy to infinity along the channel. These non-radiating antisymmetric multipoles were used by Callan, Linton and Evans (1991) and McIver (1991) to construct trapped wave solutions for a cylinder on the centre-line of a channel.

For the solution of the cylinder scattering problem it is necessary to expand the multi- pole potentials in terms of the polar coordinates r and θ. The identity

$$e^{\frac{1}{2}Z(T-T^{-1})} = \sum_{m=-\infty}^{\infty} T^m J_m(z) \tag{3.18}$$

(Abramowitz and Stegun (1965)) is used where J_m is the Bessel function of first kind of order m. Substituting the values $Z = \lambda r$ and $T = -ie^{r\pm i\theta}$ represents

$$e^{-i\lambda xt\pm\gamma\lambda(z-d)} = \sum_{m=-\infty}^{\infty}\epsilon_m(-i)^m(\cosh m\tau\,\cos m\theta \pm i\sinh m\tau\sin m\theta)J_m(\lambda r), \tag{3.19}$$

where ϵ_m is defined by the equation (2.9). Using (3.19) into (3.7) and (3.15) resulting expressions become

$$\phi_n = H_n(\lambda r)\cos n\theta + \sum_{m=0}^{\infty}(\alpha_{nm}\cos m\theta + \beta_{nm}\sin m\theta)J_m(\lambda r) \tag{3.20}$$

where

$$\alpha_{nm} = \frac{\epsilon_m(-i)^m i^{n-1}}{\pi}\int_{-\infty}^{\infty}\frac{e^{-2\gamma\mu}+\cosh 2\gamma\nu}{\gamma\sinh 2\gamma\mu}\cosh m\tau\cosh n\tau\, dt \tag{3.21}$$

$$\beta_{nm} = \frac{\epsilon_m(-i)^m i^n}{\pi}\int_{-\infty}^{\infty}\frac{\sinh 2\gamma\nu}{\gamma\sinh 2\gamma\mu}\sinh m\tau\cosh n\tau\, dt \tag{3.22}$$

and

$$\psi_n = H_n(\lambda r)\sin n\theta + \sum_{m=0}^{\infty}(a_{nm}\cos m\theta + b_{nm}\sin m\theta)J_m(\lambda r) \tag{3.23}$$

where

$$a_{nm} = -\frac{\epsilon_m(-i)^m i^n}{\pi}\int_{-\infty}^{\infty}\frac{\sinh 2\gamma\nu}{\gamma\sinh 2\gamma\mu}\cosh m\tau\sinh n\tau\, dt \tag{3.24}$$

$$b_{nm} = \frac{\epsilon_m(-i)^m i^{n+1}}{\pi}\int_{-\infty}^{\infty}\frac{e^{-2\gamma\mu}-\cosh 2\gamma\nu}{\gamma\sinh 2\gamma\mu}\sinh m\tau\sinh n\tau\, dt \tag{3.25}$$

Here, it can be seen that $\beta_{n0} = b_{n0} = 0$ for all n. The equations (3.20) and (3.23) are valid for $0 < r < 2(b-d)$. In particular, if choose $D = 0$ and $\epsilon = 0$, then the ice-cover surface reduces to a free surface and the channel multipoles exactly coincide with the channel multipoles for the case of uniform finite depth water with free surface (cf. McIver and Bennet (1993)).

International Journal of Recent Development in Engineering and Technology
Website: www.ijrdet.com (ISSN 2347-6435(Online) Volume 14, Issue 12, December 2025)

IV. Conclusion

The problem of scattering of water waves by axisymmetric bodies in a channel with uniform finite depth water beneath the free surface is extended here when the free surface is replaced by a thin ice-cover modelled as a thin elastic plate. The channel multipole potentials in a uniform finite depth channel with ice-cover are constructed in a systematic manner. Also, appropriate form of the channel multipoles can be made in the circumstances when the fluids are two-layer. In particular, When the flexural rigidity and surface density of ice-cover are taken to be zero, the ice-cover becomes a free-surface. Then the results exactly coincide with the channel multipoles for the case of uniform finite depth water with free surface.

REFERENCES

[1] Abramowitz, M. and Stegun, I.A., Handbook of Mathematical Functions. *New York: Dover*, (1965).

[2] Callan, M., Linton, C.M. and Evans, D.V., Trapped waves in two-dimensional waveg- uides, *J. Fluid Mech.*, 229 (1991) 51-64.

[3] Linton, C. M. and Evans, D.V., The radiation and scattering of surface waves by a vertical circular cylinder in a channel, *Phil. Trans. R. Soc. Lond.*, A 338 (1992) 325-357.

[4] Mclver, P., Trapping of surface water waves by fixed bodies in a channel, *Q. Ji Mech. Appl. Math.*, 44 (1991) 193-208.

[5] McIver, P. and Bennett, G.S., Scattering of water waves by axisymmetric bodies in a channel, *Journal of Engineering Mathematics*, 27 (1993) 1-29.

[6] Thorne, R. C., Multipole expansions in the theory of surface waves, *Proc. Camb. Phill. Soc.* 49(1953) 707-716.

[7] Twersky, V., On scattering by the infinite grating of circular cylinders. *IRE Trans. Antennas Propag.* 10 (1962) 737-765.

[8] Ursell,F., Surface wave on deep water in the presence of submerged circular cylinder I and II, *Proc. Camb. Phil. Soc.* 46 (1950) 141-155.

[9] Yeung, R.W. and Sphaier, S.H., Wave-interference effects on a floating body in a towing tank, *Proc. PRADs '89, Varna, Bulgaria*, (1989).