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Essentially Dense Connected Spaces

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Abstract- **In this paper we have introduced essentially dense connected spaces. We studied the structure of dense connected space and essentially dense connected spaces and their behaviour under mappings and products. We proved that the hereditarily quotient maps preserve dense connectedness and essentially dense connectedness. We have also proved that every connected subspace of an essentially dense connected space with atleast two points must have non-empty interior. Examples are provided to illustrate dc-hereditarily quotient maps and edc-hereditarily quotient maps.**

Keywords- **Continuum point, dc-hereditarily quotient map, DC-Spaces, edc- hereditarily quotient map, Essentially Dense Connected Space (EDC), Hereditarily quotient map.**

I. INTRODUCTION

Connectedness in topology is often studied through various strengthened or weakened forms that capture how points and subsets behave under closures, expansions and mappings. Among these generalization, dense connected space offers a viewpoint based on the existence of continuum points. Thus the point lying in the closure of every non-empty subset. A stronger notion, called essentially dense connectedness arise when one requires that any subspace containing a continuum point must remain connected under every expansion. This condition ensures that connectedness is preserved in very stable manner, even when new points or sets are added. Thus EDC-spaces act a robust versions of DC-spaces. Mappings also play an important role in understanding such structures. In particular, hereditarily quotient maps, which remain quotient under all restrictions, are natural tools for transferring closure and connectedness properties from one space to another. Because DC-space and EDC-space depend strongly on closure behaviour, hereditarily quotient maps form an ideal setting for studying the properties of these spaces under images and subspaces.

II. PRELIMINARIES

Definition 2.1: [2] Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set S. If $\mathcal{T}' \supset \mathcal{T}$ we say tat $\mathcal{T}'$ is finer than $\mathcal{T}$.

Definition 2.2: [2] A topology finer than a topology τ is called an expansion of τ and space S furnished with such a finer topology is called an expansion of (S, τ).

Definition 2.3: [2] A space S whose connected subsets remain connected as subspaces of every expansion in which S remains connected. Such spaces are called essentially connected.

Definition 2.4: [2] A connected space having no connected expansion is maximally connected.

Definition 2.5: [2] We call a space and its topology strongly connected if it has a maximally connected expansion.

Definition 2.6: [6] Let $f: S \to T$ is said to be restriction of f to A if the function $f|_A: A \to T$ defined by the rule $f|_A(s) = f(s)$ for all $s \in A$. It is called a restriction map.

Example 2.7: Let $f: \mathbb{R} \to \mathbb{R}, f(a) = a^2$. Consider $A = [0,2] \subseteq \mathbb{R}$ then $f|_{[0,2]}: [0,2] \to \mathbb{R}$, $f|_{[0,2]}(a) = a^2$. Hence $f(a) = \{a^2: a \in [0,2]\} = [0,4]$. Therefore $f|_A$ is continuous. In fact it is homeomorphic, $[0,2] \cong [0,4]$ with inverse $b \to \sqrt{b}$.

Definition 2.8: [6] Let S and T be topological spaces. Let $p: S \to T$ be a surjective map. The map p is said to be a quotient map provided a subset U of T is open in T if and only if $p^{-1}(U)$ is open in S.

Definition 2.9: [2] A surjective map $f: S \to T$ is hereditarily quotient if for every subset $A \subseteq S$, the restricted map $f|_A: A \to f(A)$ is a quotient map, where A has the subspace topology from S and f(A) has the subspace topology from T.

Example 2.10: Let $S = T \times \{0\}$ *define* $\pi: S \to T$ by $\pi(t,0) = t$ *and* $\varphi: T \to S$ by $\varphi(t) = (t,0)$. Then for every $t \in T, (\pi \circ \varphi)(t) = \pi(t,0) = t, (\varphi \circ \pi)(t,0) = \varphi(\pi(t,0)) = \varphi(t) = (t,0)$. It is a homeomorphism. Let $A \subseteq S$. Consider the restriction $\pi|_A: A \to \pi(A)$. The inverse of this restriction is $\pi|_{\pi(A)}$, and since both $\pi|_A$ and $\pi^{-1}|_{\pi(A)}$ are continuous. Here $\pi|A$ is homeomorphism onto (a) . In particular, $\pi|_A$ is a quotient map. Since A is arbitrary, π is hereditarily quotient map.

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Definition 2.11: [6] A subset C of S is saturated (with respect to the surjective map $p: S \to T$) if C contains every set $p^{-1}(\{y\})$ that it intersects. Thus C is saturated if it equals the complete inverse image of a subset of T.

III. Essentially Dense Connected Spaces

Definition 3.1: Let S be dense connected and Let T be the subspace of S with continuum point. If every expansion of T remains connected then S is said to be essentially dense connected.

Example 3.2: Let $f: \mathbb{R} \to \mathbb{R} \times 0$, defined by $f(a) = (a, 0)$ be a homeomorphism. Let $A \subseteq \mathbb{R}$ be a connected subspace with atleast two points. Any such interval contains a non-empty open interval and $\text{int}(A) \neq \emptyset$. Therefore, $\mathbb{R}$ is a EDC −Space. Hence $\mathbb{R} \times 0$ is a EDC −Space.

Example 3.3: Let the Sierpinski Space $S = \{0,1\}$, then $\tau = \{\emptyset, \{1\}, S\}$ is a Dc −Space.

Example 3.4: Let $S = \{\left(s, \sin\left(\frac{1}{s}\right) : s > 0\right\} \cup (\{0\} \times -1,1\}$ be a connected set. Every non-empty subset has a closure point and it is dense connected. But some connected subsets with atmost two points have empty interior. Therefore the set is DC −space but not EDC −space.

Theorem 3.5: Let (S, τ) be an essentially dense connected space and let C be a connected subspace having atleast two points then $\text{Int}(C) \neq \emptyset$.

Proof: Let S be essentially dense connected space. Let C be a connected set with atleast two distinct points c_1, c_2 and $c_1 \neq c$. Consider the set T contained in C and T has a continuum point t_1. Clearly $t_1 \in \overline{T}$. Here $T \subseteq C \subseteq S$, then T is an expansion of τ. Assume that $\text{int}(C) = \emptyset$, then all points of C lie in the boundary of C, it is possible to pick a point $s \in S \setminus C$.Now consider the expansion $C \cup \{s\}$. If $s \in \overline{C}$, then the singleton $\{s\}$ is an open subset of $C \cup \{s\}$. That is $\{s\}$ disconnects the expanded set C and $\{s\}$. By the definition of essentially dense connected space, every expansion of τ must remains connected, which is a contradiction. Therefore, $\text{int}(C) \neq \emptyset$.

Theorem 3.6: Let (S, τ) be an essentially dense connected space, and Let C be a connected subspace having atleast two points then (S, τ) is a dense connected space.

Proof: Let (S, τ) be an essentially dense connected space. Take any point $s \in S$. The singleton $\{s\}$ satisfies $s \in \overline{S}$ and has a continuum point. By the definition of essential dense connected space every expansion of S is connected. Hence S is connected. If there exist non-empty disjoint sets $E, F \subseteq A$ that are open in the subspace A then $A = E \cup F$. Hence $\overline{E}$ and $\overline{F}$ are closed, non-empty and $\overline{E} \cup \overline{F} = \overline{A} = S$ where each of $\overline{E}$ and $\overline{F}$ is connected, then $\overline{E}$ and $\overline{F}$ are expansions of singletons from E and F respectively. Since S is connected and $S = \overline{E} \cup \overline{F}$ is the union of two connected closed sets, their intersection must be non-empty. By the definition of essential dense connected space, any expansion of E that contains the point $\{e\} \in E$ is connected. Thus $E \cup \{e\}$ is connected. Similarly $F \cup \{e\}$ is connected. Because $(E \cup \{e\})$ and $(F \cup \{e\})$ are connected their union $(E \cup \{e\}) \cup (F \cup \{e\}) = A \cup \{e\}$. A is disconnected by removing the point $\{e\}$. The set $A = (A \cup \{e\})\{e\}$ is obtained from a connected set $A \cup \{e\}$ by deleting one point. By the removal of point $\{e\}$ there exists two disjoint non-empty open sets in A with the subset E, F. This contradicts the assumption that A is disconnected. Hence A must be connected. Every non-empty dense subset A of S is connected. Hence an essentially dense connected space S is dense connected.

Theorem 3.7: Let (S, τ)be an essentially dense connected space and let $A, B \subseteq S$ be connected subsets. Let $A \cap B = \bigcup_{i \in I} C_i$ then every component C_i of $A \cap B$ that contains atleast two points has non-empty interior in S.

Proof: Let C_i be a component of $A \cap B$ with atleast two points. Then C_i is a non-degenerate connected subspace of S. By the definition of essentially dense connected space, every non-degenerate connected subset of S must have non-empty interior in S. Hence $\text{int}_S(C_i) \neq \emptyset$. Therefore every component C_i of $A \cap B$ that contains atleast two points has non-empty interior in essentially dense connected space S.

Theorem 3.8: Let (S, τ) be an essentially dense connected space and let A, B be connected subsets of S. Let $A \cap B = \bigcup_{i \in I} C_i$. If $A \cap B$ has atleast two components containing more than two points then the family C_i, where $i \in I$, cannot be closure-preserving.

Proof: Suppose $A \cap B$ has two disjoint components with more than two points $C_i, C_j, i \neq j$. By the above theorem, both C_i and C_j contain non-empty open subsets of S.

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Hence $\mathrm{int}(\overline{C_i}) \neq \emptyset, \mathrm{int}(\overline{C_j}) \neq \emptyset$. Since $A \cap B \subseteq A$ and A is connected, they must intersect each other. Thus $\overline{C_i} \cap \overline{C_j} \neq \emptyset$. Now consider the closure of their union $\overline{C_i \cup C_j}$. Because the closures intersect, $\overline{C_i} \cup \overline{C_j} = \overline{C_i \cup C_j}$. A closure preserving family C_i must satisfy the condition $\overline{\bigcup_i C_i} = \bigcup_i \overline{C_i}$. However, because two components have intersecting closures, the family cannot be closure-preserving where more components are added. Thus C_i is not closure-preserving and $A \cap B$ has non-empty interior in EDC −space S.

Example 3.9: Let $A = [0, 2]$, $B = [1, 3]$ in $\mathbb{R}$. Then $A \cap B = [1, 2]$. Consider $B = [1, 1.15] \cup [1.6, 3]$, then $A \cap B = [1, 1.5] \cup [1.6, 2]$ which is disconnected. If the components are $C_1 = [1, 1.5], C_2 = [1.6, 2]$ then $\overline{C_1 \cup C_2} = [1, 2] \neq C_1 \cup C_2$. Here the connection fails to be closure-preserving.

Theorem 3.10: Let $S = \prod_{i\in I} S_i$. Then the product space S is essentially dense connected if and only if exactly one factor is essentially dense connected and every other factor is a singleton.

Proof: Assume $S = \prod_{i\in I} S_i$. Suppose that two factors S_p and S_q have atleast two distinct points in each, $a_p \neq b_p \in S_p$ and $a_q \neq b_q \in S_q$. Fix any points in all other co-ordinates. Consider the diagonal like connected set. $C = \{(S_p, S_q): S_p = a_p + (t), S_q = a_q + (t) \mid t \in [0,1]\}$. This set C is connected, non-degenerate but has empty interior in the product. Thus the product is not essentially dense connected, which is a contradiction. Hence, atmost one factor has more than one point. Conversely, assume S_i is the only non-singleton factor. Then every other factor is a single point $S = S_j * \prod_{i\neq j}\{s\}$. The product with a one point space is homeomorphic to the factor S_j. Since essential dense connectedness is preserved by homeomorphism, S_i must also be essentially dense connected space. If only one factor S_i has atleast two points and all other are points, then, $S = S_j * \prod_{i\neq j}\{s\} \cong S_j$. A homeomorphic image of an essentially dense connected space is essentially dense connected space. Hence the product is essentially dense connected space.

IV. Hereditarily Quotient Maps

Definition 4.1: A surjective map $f: S \to T$ is said to be dense-connected hereditarily quotient (dc-hereditarily quotient) if for every subspace $A \subseteq S$, the restricted map $f|_A: A \to f(A)$ is a quotient map and if A is dense connected in S then f(A) is dense connected in T.

Example 4.2: Consider $f: \mathbb{R}^2 \to \mathbb{R}$, defined by $f(a, b) = a$. Let every $A \subseteq \mathbb{R}^2$ the restriction $f|_A: A \to f(A)$ is a quotient map. If A is connected and $A \subseteq \mathbb{R}^2$, then f(A) is connected and $\overline{f(A)} = \mathbb{R}, \overline{f(A)} \supseteq f(\overline{A}) = f(\mathbb{R}^2) = \mathbb{R}$. So f(A) is dense connected in $\mathbb{R}$.

Definition 4.3: A surjective map $f: S \to T$ is called essentially dense connected hereditarily quotient (edc-hereditarily quotient) if for every subset $A \subseteq S$, the restricted map $f|_A: A \to f(A)$ is a quotient map and if A has a continuum point and every expansion of A in S is connected, then every expansion of f(A) in T is also connected.

Remark 4.4:

1. A hereditarily quotient map related to dense connectedness is a surjective map whose restriction is a quotient map and which preserves dense connectedness.
2. A hereditarily quotient map related to essentially dense connectedness is a surjective map whose restrictions are quotient maps and preserves the connectedness of all expansions of sets with continuum points.

Theorem 4.5: If $f: S \to T$ is a dc-hereditarily quotient and S is a DC-space, then T is DC-space.

Proof: Consider $f: S \to T$ is a dc-hereditarily quotient map. Let B be a non-empty subset of T. Assume that A is contained in S with $f(A) = B$ where $f^{-1}(B) = A$. Since S is a DC-space, A has a continuum point. Thus $B = f(A)$ is a DC-space in T. Hence every non-empty subset of T has a continuum point, and T is a DC-space.

Theorem 4.6: If $f: S \to T$ is a edc-hereditarily quotient map and S is EDC-space then T is EDC-space.

Proof: Let B be a subset of T and B has a continuum point. Assume $A \subseteq S$ with $A = f^{-1}(B)$. Because f is edc-hereditarily quotient map, expansions of A that are connected map to expansions of B that remain connected. By the definition of EDC-space all expansions of A are connected. Therefore all expansions of B are connected in T. Hence T is EDC-space.

Theorem 4.7: Let $f: S \to T$ and $g: T \to U$ be surjective maps. Then the following conditions are true

i. If f and g are hereditarily quotient map then $g \circ f$ is hereditarily quotient map.

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ii. If f and g are dc- hereditarily quotient map then $g \circ f$ is dc-hereditarily quotient map.

iii. If f and g are edc- hereditarily quotient map then $g \circ f$ is edc- hereditarily quotient map.

Proof:

i. Let A be a subset of S. Let f, g be quotient maps and $g|_{f(A)}: f(A) \rightarrow g(f(A))$ is quotient map then $f(A) \subseteq T$. The composition of quotient maps is quotient map. Therefore, $(g \circ f)|_{f(A)} \circ f|_A$ is a quotient map of $A \rightarrow g\big(f(A)\big)$. Since A is arbitrary and $g \circ f$ is a quotient map, then $g \circ f$ is a hereditarily quotient map.

ii. Let f and g be dc- hereditarily quotient maps. Then for any A contained in S, the image is $(g \circ f)(A) = g\big(f(A)\big)$. Also restriction quotient by (1) remains same. Hence $g \circ f$ is dc- hereditarily quotient map.

iii. Simillarly, the proof is obvious.

Theorem 4.8: Let $f: S \rightarrow T$ be hereditarily quotient map. *Then,*

i. For every $A \subseteq S$ the restricted map $f|_A: A \rightarrow f(A)$ is a quotient map.

ii. If f is dc- hereditarily quotient map and $A \subseteq S$ is a DC-space as a subset of S, then f(A) is a DC-space which is a subset of T.

Proof:

i. The proof is obvious by the definition.

ii. Let $A \subseteq S$ be a DC-space. Since f is dc-hereditarily quotient map, the image $f(A) \subseteq T$ is the DC-space by the definition of dc- hereditarily quotient map

Remark 4.9: The above theorem is true for the $edc-$ hereditarily quotient map condition.

Theorem 4.10: If $f_i: S_i \rightarrow T_i, i \in I$ be a family of surjective continuous maps and let $F = \prod_{i \in I} f_i : \prod_i S_i \rightarrow \prod_i T_i$ denote the product map then *F* is hereditarily quotient map.

Proof: Let $f_i: S_i \rightarrow T_i, i \in I$ be a family of surjective continuous maps. If each f_i is open or closed, then F is open or closed where F is hereditarily quotient map. Consequently, if each f_i is dc- hereditarily quotient map and each f_i is open or closed then F is $dc-$hereditarily quotient map.

In open case, let U be the open set in $\prod_i S_i$, say $U = \prod_i U_i$ where each $U_i \subseteq S_i$ is open and $U_i = S_i$ for all but finitely many i then $F(U) = \prod_i f_i(U_i)$. In the closed case the proof is obvious. Thus F is closed and therefore hereditarily quotient map.

Remark 4.11:

1. The product of arbitrary quotient maps need not be a quotient map.

2. Any homeomorphism is hereditarily quotient map, hence trivially $dc-$hereditarily quotient map and $edc-$ hereditarily quotient map.

Theorem 4.12: Let $f: S \rightarrow T$ be a hereditarily quotient map. Let $A \subseteq S$ be any subset of S, and $f|_A: A \rightarrow f(A)$. Then the following conditions are true

i. If A is saturated with respect to $f, A = f^{-1}\big(f(A)\big)$, then $f|_A$ is also a quotient map.

ii. If f is hereditarily quotient map, then $f|_A$ is hereditarily quotient map.

Proof: Consider A is saturated, for any open set U and V where $U \subseteq A, V \subseteq S, U = A \cap V$. For a subset $W \subseteq f(A), (f|_A)^{-1}(W) = A \cap f^{-1}(W)$. Since f is a quotient map, $f^{-1}(W)$ is open in S iff W is open in f(S). Therefore $(f|_A)^{-1}(W) = A \cap f^{-1}(W)$. Hence W is open in f(A) iff its preimage is open in A. So $f|_A$ is a quotient map. Now let $B \subseteq f(A)$. The map $(f|_A)|_{(f|_A)^{-1}(B)}$ which is $f|_{(f)^{-1}(B)}$ is a quotient map. So $f|_A$ is a hereditarily quotient map.

Theorem 4.13: Let $f: S \rightarrow T$ be a hereditarily quotient map and let S be a DC-space, then T is also a DC-space.

Proof: Let $B \subseteq T$ be non-empty subset of T. Consider the preimage $f^{-1}(B) \subseteq S$. Since $f^{-1}(B) \neq \emptyset$, and S is a DC-space, it contains a continuum point, say $s \in cl_s\left(f\left(f^{-1}(B)\right)\right) = cl_T(B)$. Thus every non-empty set $B \subseteq T$ has a continuum point. Hence T is dense connected space.

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