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Mathematical Modelling of Effect of Green House Gases on Polar Ice Melting

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***Abstract*— The atmospheric concentrations of greenhouse gases, such as carbon dioxide (CO_2), methane (CH_4), and nitrous oxide (N_2O) have a significant increase in the past 150 years. Carbon dioxide forces the earth's energy out of balance by absorbing thermal infrared energy (heat) radiated by the surface, causing the ocean's surface to warm, and melting more and more polar icecaps. Since polar icecaps help to regulate the earth's climate system, the fate of the Arctic icecaps is critical to the future climate. This paper exhibits different mathematical models to predict rate of ice-depletion in Arctic Ocean, and to analyse the trend of the Arctic icecap. The prediction results shows that Artic ice cap will be free of ice by the year 2035.**

***Keywords*— Atmospheric concentrations, Greenhouse gases, moving boundary problems, mathematical modelling, sea ice.**

I. INTRODUCTION

The dynamics of arctic sea ice melting is a complex phenomenon, that requires better understanding and multiple variables to describe the process. Atmospheric concentration of carbon dioxide has increased to unprecedented level by 40% since pre-industrial times [1]. Sea ice forms and melts in sea-water, this can be seen at the polar icecaps located at the arctic and antarctic poles. The polar icecaps help to regulate the global temperature, and play an important role in the global climate system. If the arctic ocean is going to be ice free, it will have multiple impacts. It will cause global temperature to rise, and accelerate the melting of the global ice sheets, which hold enough water to raise sea levels. Since sea ice can efficiently reflect the sun's radiance, sea ice plays a crucial role in the global ocean circulation and regulation of climate. Area, thickness and age of sea ice in the arctic sea is the proxy of earth's temperature. Since the 1970s, satellites have provided the best method to allow scientists to monitor the inter-annual variations and trends in sea ice cover.

Scientists can collect data of arctic sea ice during specific times to calculate and observe the changes in global temperature and climate. In recent years, satellite data shows the arctic sea ice having a shrinking trend; more sea ice is melting into the seawater due to the increasing temperature of the ocean's surface. In figure 1, the magenta line shows the 1981 to 2010 median extent for that month. The black cross indicates the geographic north pole.

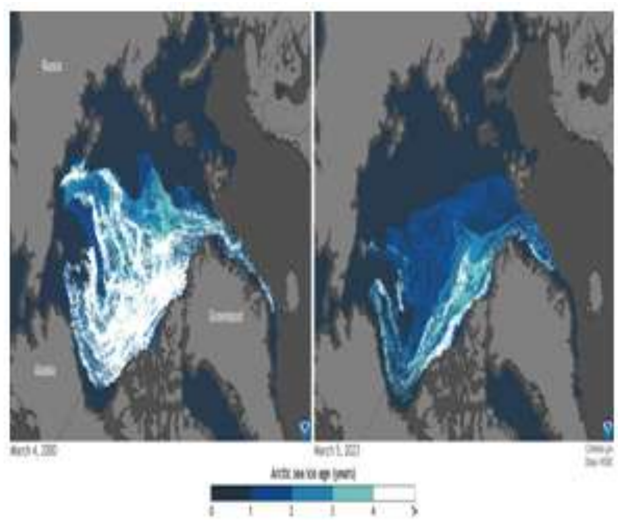

Figure 1.1: Sea ice age, the week of March 4, 2000 (left) and the week of March 5, 2023 (right).

Oldest sea ice is white, and youngest sea ice is dark blue. The extent of old, thick sea ice in the Arctic has declined significantly since the mid-1980s, when satellite measurements first became available, and even since the start of the twenty-first century.

"Data source: NOAA Climate.gov, adapted from the National Snow and Ice Data Centre (NSIDC)."

II. METHODOLOGY

Mathematical models are essential for analysing how increasing atmospheric carbon dioxide affects the melting behaviour of polar ice. To represent these complex processes, researchers use tools ranging from simple linear regression analyses to advance nonlinear simulation frameworks. Most of the physical phenomenon that takes place in sea ice modelling is highly non-linear. To describe the non-linear behaviour of these real-world problems, and how specific variables change both spatially and temporally in relation to each other partial differential equations come into existence. These equations explain the diverse phenomenon of phase change problems that occur in energy conservation units, thermal engineering, weather forecasting etc. These partial differential equations are computationally simulated to explain the key variables and how these variables explain the changes in the pattern of the solution. For complex phenomena, the equations involve multiple variables and get complicated very quickly. Modelling the sea ice, is a complex phenomenon that involve multiple variables that represent the thickness of the ice, density of the ice, concentration of ice in an area, the movement of ice etc. Also, there are many external factors that affect the process like wind speed, temperature. Analytical solutions for these problems exist for simple geometries and neglecting many of the external forces. Whereas numerical methods are helpful to solve for different geometries and play a crucial role in simulating and studying ice melting in polar-regions. These methods help to solve the complex equations and computationally simulate the behaviour. The objective of the present study is to build a mathematical modelling for different cases of increasing complexity, and to verify the effect of the process parameters on the depletion of sea ice. Further R-programming software is used to find the trend and the correlation for getting the results. The paper is discussed in two different cases with different solution techniques.

Model 1: Developing a model that describe the Arctic icecap extent and the CO_2 atmospheric concentration, using a linear regression model.

Model 2: Develops an advanced exponential model, briefly introduce how to test whether the regression model is proper for the dataset, and makes a prediction of the Arctic open water area as a function of time, by using the normal distribution probability density function. Summarizes the prediction results and its impacts in the future.

III. LITERATURE REVIEW

Since most of the real-world problems are non-linear in nature, Scientific computations and engineering problems are better explained using Partial differential equations. Phase change problems are non-linear problems, therefore, very few analytical solutions are available in the literature [3]. These problems have many industrial and technological implementations, due to their wide range of applications such as the melting of ice in the polar- regions, determination of the depth of the frost or thaw penetration, energy storage techniques, etc. tracking the interface position and the rate at which the interface moves are important parts of the solution, as there exists a moving interphase through-out the process. These problems are addressed by approximation, semi analytical or by numerical methods. In recent times more advanced models of sea ice depletion are studied as these problems play a major role in the effect of greenhouse gases on the environment. Different studies explain how key parameters affect the polar ice melting and decrease in the depth of the ice.

Models that involve atmospheric changes are complex to understand the phenomenon. Sea-level (SL) rise basically depend on the complex interactions between ice-sheets, icebergs, ocean and the atmosphere. A thermodynamic sea-ice model which is modified by Fang et al. [4], using a three-layer structure. The study focuses on snow heat capacity, vertical salinity gradients, and temperature-dependent conductivity in the sea. Due to the considerations of three-layer model with significant changes compared with original Winton's model, the model accounts that there is an increase in the ice thickness. Experiments were conducted using the Modular Ocean Model version 4 (MOM4) coupled with the modified three-layer model. The results show an improved sea ice simulation with an increase in both the sea ice volume and thickness over the entire Arctic region.

Jun-Young Park [5], worked on Antarctic sea-level rise and the effect of various interactions with the ice sheets. The objective of the paper is to quantify the effect of the interactions. Global climate-ice-sheet models that capture the complex interactions between climate components and ice-sheet, ice-shelf and iceberg dynamics and thermodynamics are incorporated in the study. This study focuses on the Antarctic Ice Sheet (AIS) and Greenland ice-sheet (GrIS) contributions to future Sea Level (SL) projections.

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Numerical simulations conducted with the coupled three-layer climate-ice-sheet-iceberg modelling systems. The formation of ice in the open water areas on the edge of existing sea ice has a significant effect on polar atmosphere, ocean, and ice interactions.

Fanghua Wu et al. [6] carried simulation of Antarctic Sea ice concentration and thickness using a coupled climate model BCC- CSM2- MR. In this study, a modified collection depth parameterization of new ice based on an existing scheme is presented. Also, modification of the sea ice concentration and ice growth rate are added as additional factors. This study reveals that it improves simulation of Antarctic Sea ice concentration and thickness in most of Indian and Atlantic sectors. Furter, the study exposes that these improvements are due to the early closure of open water areas in model simulations.

Using the non-linear growth theory in a semi-enclosed and seasonally ice-covered Kara Sea as a systematic case study by Chenglin Duan et al. [7], focuses on the theoretical mathematical expressions of the nonlinear freezing and melting processes of sea ice. In this case study periodic cycle of sea ice coverage was divided into freezing, frozen, and melting stages, with the continuous ice-coverage curves clearly highlighting temporal nonlinearity during both freezing and melting periods. Results showed the Logistic function to most accurately quantify the annual periodic cycle, in comparison with various functions.

Modelling of global warming effect on the melting of polar ice caps with optimal control analysis was studied by Andry Dwi Kurniawan et al. [8]. The study focuses on the effect of increasing carbon dioxide gas emissions as the growing threat of global warming and its effect on melting polar ice caps. The stability analysis of three equilibria, Human and forest absence, forest absence, and co-existence are considered as three equilibria for the study of stability analysis presented and reforestation as control variables. Pontryagin's Maximum Principle is used to solve the stability analysis problem. The simulation shows that the combination of two controls give a significant impact to reduce the concentration of carbon dioxide and the rate of melting polar ice caps.

Mathematical Modelling of Glacier Melting in the Arctic with regard to climate warming was carried by Anatoliy Fedotov et al. [9]. The model was developed taking into account the solid-liquid phase transition, that describes the temperature distribution in the glaciers. Finite volume method is applied to solve the problem numerically.

Temperature profiles were analysed for the two glaciers Vavilov Ice Cap and Austre Gronfjordbreen, for three different global-warming scenarios (RCP2.6, RCP7, and RCP1.9) projected to the year 2100. The simulations confirmed the global warming for all the three scenarios. RCP1.9 would notably slow this melting, emphasizing the urgency of controlling global temperature rise.

IV. Preliminary Model

In the late 1950s, Charles David Keeling of the Scripps Institution of Oceanography initiated continuous measurements of atmospheric CO_2 at Hawaii's Mauna Loa Observatory, chosen for its stable conditions and minimal local interference from vegetation or weather. Analysis of Mauna Loa records from 1959 to 2013 [2] shows a nearly linear rise in annual-mean atmospheric CO_2 concentration (ppm). A linear-regression model was therefore used to examine temporal trends in atmospheric CO_2 levels

$$y = a_1 + a_2 x \qquad (2.1)$$

Equation (2.1) is the linear regression model. Let y be the mean CO_2 concentrations in the given years, and x is the time measured in years. The unknown parameters a_1 and a_2 are to be determined. If a_2 is positive, that indicates the CO_2 concentration is increasing over the years. Once the unknown parameters are obtained, it is possible to compute the residuals $r = y - y1$ and R^2 which is the coefficient of determination.

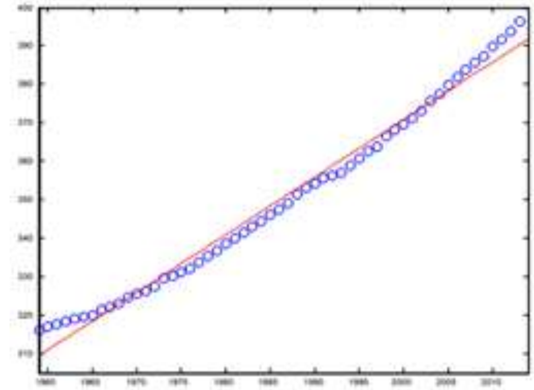

Figure 2.1: Linear-regression fit between mean atmospheric CO_2 concentration vs time.

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$$R^2 = \frac{\|r\|^2}{\|y - y_1\|} \qquad (2.2)$$

where $y = \frac{1}{n}\sum_{i=1}^{n} y_i$

In Figure 2.1, the parameters a_1 = 307.9993 and a_2 = 1.4961. Residue standard error is 2.8231, and R^2 = 0.9866. The parameter a2 = 1.4961 (positive) indicates that the CO_2 concentration at atmosphere is increasing over the past 55 years. The value of R^2 is close to 1, indicates that every variation in x with the corresponding y is well explained by this regression model. If CO_2 is considered the dominant driver of Arctic ice-extent change, the model can approximate the time of complete summer ice loss under unchanged emission conditions. The data is obtained from National Oceanic and Atmospheric Administration (NOAA), a scientific agency focusing on the oceans and the atmosphere [10]. Towards the end of the summer the extent of sea ice is maximum in the Northern Hemisphere, hence considering September months extent data is an ideal choice.

Linear Regression is used to estimate the relation between the ice-cap extent and time, because the data trend of the Arctic icecap extent has an approximate linear shape. Based on the dataset [10] and Equation (2.1), assume x = {0, 1, ……, 35}, and y is the icecap extent of North Pole in the given years.

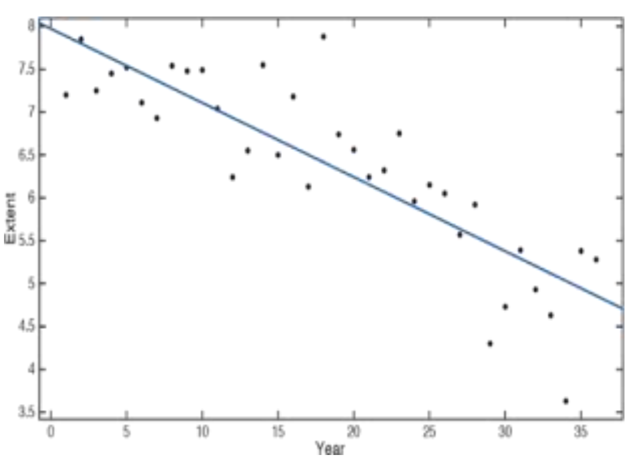

Figure 2.2: The linear regression of the Arctic icecap extent vs. time.

In Figure 2.2, the vertical axis is the Arctic Ocean icecaps extents in million square kilometres; the horizontal axis is time- period. The linear regression line provides all the parameters and residuals. The parameters a_1 = 7.972968, and a_2 = 0.086497. Residue standard error is 0.5663, and R^2 = 0.7192. The positive slope (a_2 = 0.086497) confirms a steady decline in Arctic ice extent over time.

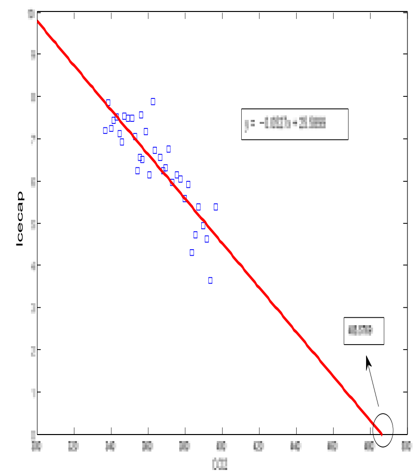

Figure 2.3: A linear regression fit of the Arctic icecap extent vs. CO_2 concentration.

Northern Hemisphere ice extent vs CO_2 concentrations were plotted and a linear regression model is prepared to fit the data to find the CO_2 concentration when the icecap extent is approximate zero. In Figure 2.3, it is shown that the North Pole would be ice-free when the CO_2 concentration is 485.5796 ppm (micromole/mol.). After that, plug this CO_2 concentration value into Figure 2.1, which is the linear regression plot of CO_2 concentration over the years. By using the correlation of the CO_2 concentration and time, the time of the Arctic Ocean being ice-free is predictable.

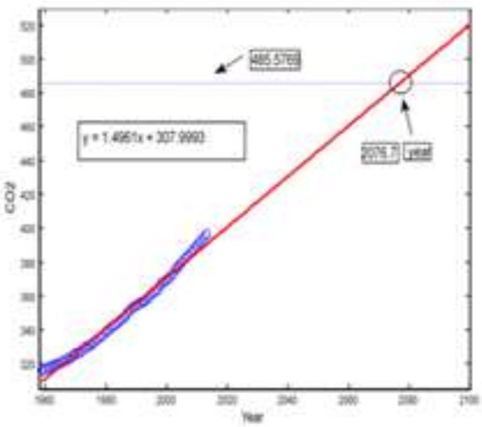

Figure 2.4: Alinear regression of the Co_2 concentration vs. time.

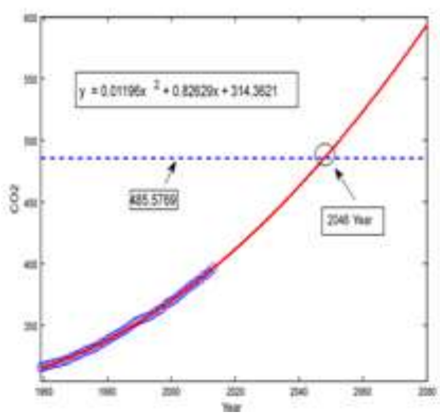

Figure 2.5: Aquadratic regression of the Co_2 concentration vs. time

The prediction result is shown in Figure 2.4. and Figure 2.5. The model projects complete Arctic ice-loss around 2076 under the assumed linear trend. Based on the behaviour of these data points, the quadratic regression could be the other model choice for the Arctic Ocean icecap prediction. Because as shown in Figure 2.2 the tail of the data line tilts upward slightly, in this situation, the quadratic regression model could provide some radian on the curve. In figure 2.5, quadratic fit predicts an earlier ice-free year, approximately 2048, which is 28 years earlier, than the linear model.

V. Advanced Model

To describe the phenomenon of extinct of ice from the polar regions, where the open water area continuously changes with time, more advanced models are considered. This scenario is better described using the ordinary differential equations, as the rate of change of open water area is proportional to the total open water area.

$$\frac{dA}{dt} = kA \dots\dots\dots (3.1)$$

A= open water area: t=time; k=parameter. This ordinary differential equation describes that there is a correlation between the expansion rate of open water area and the total open water area in the Arctic Ocean. Because open water has lower albedo than ice, it absorbs more solar radiation and retains heat, thereby accelerating further melting. As the heat energy trapped increases along with the area covered, the rate of expansion also increases.

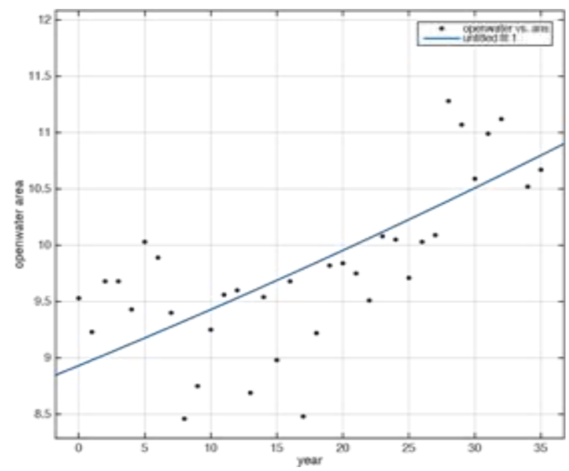

Figure 2.6: Exponential regression model
$f(x) = 8.932e^{0.005421\,t}$
for the open water area vs. time

The solution to the ordinary differential equation (3.1) is

$$A = ae^{bt} \dots\dots\dots (3.2)$$

where a = 8.932, and b = 0.005421.

The solution provides an exponential relation between the Arctic open water and the time. The data points chosen to plot the graph (2.6) are limited, so fitting an exponential curve for these data points may not be a proper choice, and it requires further investigation. Therefore, it is necessary to test this exponential regression model for more data points.

VI. An Advanced Model With CO_2

CO_2 concentration in the atmosphere is a prime key factor for the icecaps melting. A more refined advanced model includes CO_2 factor in the model equation (2.7).

$$\frac{dA}{dt} = k(at+b)A \dots\dots\dots (3.3)$$

A is the Arctic Ocean open water area, t is year, a and b are the parameters from the linear regression model, and k is a parameter to be determined. The model relates the expansion rate of open water area to the total open water area in the Arctic-sea and the concentration of CO_2 in the atmosphere. In the first case it is drawn from the results that the total open water area is directly proportional to the rate at which the area is melting. In addition, the density of greenhouse gas CO_2 acts like an insulated coat that keeps the ocean's surface warm, so that the extra heat causes more and more sea ice to melt. Therefore, the expansion rate of open water is not only closely related to the existing open water area, but it is also correlated with the concentration of CO_2 in the atmosphere. Solving equation (3.3)

$$\frac{dA}{dt} = k(at+b)A$$

$$\int \frac{1}{A} dA = \int k(at+b)dt$$

$$\log(A) = k\left(\frac{a}{2}t^2 + bt\right) + c_0$$

$$A = ce^{k\left(\frac{a}{2}t^2+bt\right)} \dots\dots\dots (3.4)$$

Equation (3.4) is a second order polynomial function with respect to the independent variable t. The parameter k and c are to be determined by taking the natural log on both sides

$$\log(A) = \log(c) + k\left(\frac{a}{2}t^2 + bt\right) \dots\dots (3.5)$$

Let $P = \left(\frac{a}{2}t^2 + bt\right)$ this is known value as a and b are known from earlier models.

$$\log(A) = \log(c) + kP \dots\dots (3.6)$$

In (3.6) A and P are known. We use polynomial regression to fix the parameters k and c.

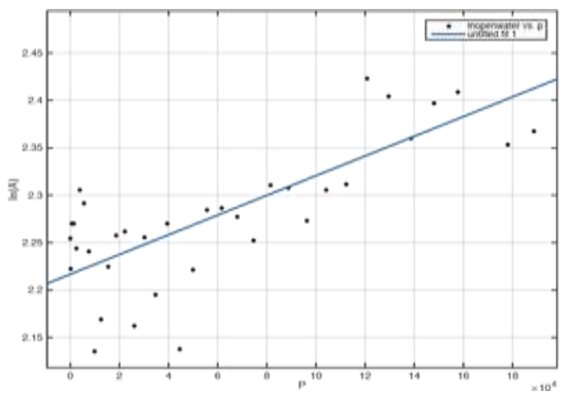

Figure 2.7: First order polynomial regression plot of $\log(A) = \log(c) + kP$

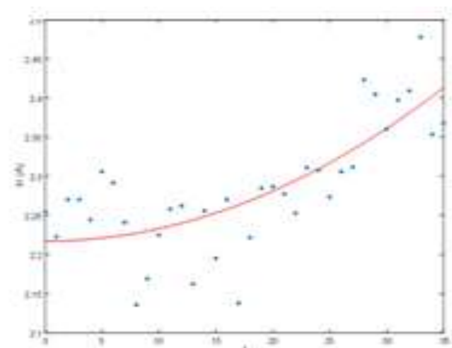

Figure 2.8: Second order polynomial regression plot of. $\log(A) = \log(c) + K_1t^2 + K_2t$

Where $K_1 = \frac{ak}{2}$ and $K_2 = bk$

The results revealed that the revised exponential regression model is more appropriate for the dataset. Furthermore, it is convincing to make a prediction of the Arctic ice caps by using this advanced model.

VII. Conclusion

The dynamics of Arctic Sea ice melting/freezing is a complex phenomenon, that requires better understanding and multiple variables to describe the process. In this work, an attempt is made to understand the process with different models, starting from the basic linear regression model to more advanced model of exponential regression modelling. The results drawn are more significant and explain the effect of CO_2 on the melting of sea ice. Using different mathematical models, quantitatively explain the annual sea ice depletion rate. In this study, the atmospheric CO_2 concentration is assumed the only factor affecting the rate of the Arctic Sea icecap melting. For prediction of Earth climate system, need to incorporate factors that describe the system more comprehensively. This will be carried out in the future study.

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